Decomposition horizons and tame graph classes

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Abstract

Low treedepth decompositions are central to the structural characterizations of bounded expansion classes and nowhere dense classes, and the core of main algorithmic properties of these classes, including fixed-parameter (quasi) linear-time algorithms checking whether a fixed graph F is an induced subgraph of the input graph G. These decompositions have been extended to structurally bounded expansion classes and structurally nowhere dense classes, where low treedepth decompositions are replaced by low shrubdepth decompositions. In the emerging framework of a structural graph theory for hereditary classes of structures based on tools from model theory, it is natural to ask how these decompositions behave with the fundamental model theoretical notions of dependence (alias NIP) and stability.

Our first main result proves that the model theoretical notions of NIP and stable classes are transported by decompositions. Precisely: Let $\mathscr{C}$ be a hereditary class of graphs. Assume that for every p there is a hereditary NIP class $\mathscr{D}_p$ with the property that the vertex set of every graph $G \in \mathscr{C}$ can be partitioned into $N_p = N_p(G)$ parts in such a way that the union of any p parts induce a subgraph in $\mathscr{D}_p$ and $\log N_p(G) \in o(\log |G|)$. We prove that then $\mathscr{C}$ is (monadically) NIP. Similarly, if every $\mathscr{D}_p$ is stable, then $\mathscr{C}$ is (monadically) stable. Results of this type lead to the definition of decomposition horizons as closure operators. We establish some of their basic properties and provide several further examples of decomposition horizons.

Our second main result establishes that every stable hereditary graph class can be decomposed in such a manner into the much simpler classes of bounded shrubdepth, generalizing the initial result concerning low treedepth decompositions of nowhere dense classes.

1 Introduction and Previous Work

In the late 90's, Baker [2] introduced the shifting strategy, allowing a linear time approximation scheme for independent sets on planar graphs. The idea is to start a breadth-first search at a vertex v of a planar graph, which partitions the vertex set of the graph into layers $L_1, \dots, L_h$ and to fix an integer D. Then, for given $s \in [D]$, by deleting all the layers L_i with $i \equiv s \bmod D$, one gets a graph with treewidth bounded by $3D$, on which a maximum independent set can be found in linear time. Considering all the possible values of s, we obtain a $(1 + 1/D)$-approximate solution of the problem. Note that grouping the layers L_i with i in a same class modulo D yields a partition of the vertex set into D parts $V_0, \dots, V_{D-1}$ such that the union of any $p < D$ of them induces a subgraph with treewidth at most $3p + 4$.

This approach was further developed by DeVos et al. [7], who proved in particular that for every proper minor closed class of graphs $\mathscr{C}$ and every integer p, there exists an integer N_p such that the vertex set of every graph $G \in \mathscr{C}$ can be partitioned into N_p parts, each p of them inducing a subgraph with treewidth at most $p-1$.

This result has been further extended by two of the authors of the present paper in a characterization of both bounded expansion classes and nowhere dense classes. Before stating these results, recall that the *treedepth* of a graph G is the minimum depth of a rooted forest F, such that G is a subgraph of the closure of F (the graph obtained from F by adding edges between each vertex and its ancestors). With this definition, the characterization theorems read as follows.

Theorem 1.1 ([15]). *A class $\mathscr{C}$ has bounded expansion if and only if, for every parameter p, there is an integer N_p such that the vertex set of each graph $G \in \mathscr{C}$ can be partitioned into at most N_p parts, each p of them inducing a subgraph with treedepth at most p.*

Theorem 1.2 (see [16,17]). *A class $\mathscr{C}$ is nowhere dense if and only if, for every parameter p and for every graph $G \in \mathscr{C}$ there is an integer $N_p(G) \in |G|^{o(1)}$, such that the vertex set of G can be partitioned into at most $N_p(G)$ parts, each p of them inducing a subgraph with treedepth at most p.*

The notions of classes with bounded expansion and of nowhere dense classes are central to the study of classes of sparse graphs [16]. Note that the treewidth of a graph is bounded from above by its treedepth and hence by the result of DeVos et al. [7] and Theorem 1.1 every proper minor closed class has bounded expansion. Surprisingly, it appeared that for monotone classes of graphs, the notion of nowhere dense class of graphs coincides with fundamental dividing lines introduced in modern model theory [21]:

Theorem 1.3 ([1]). *For a monotone class of graphs $\mathscr{C}$, the following are equivalent:*

(1) $\mathscr{C}$ is nowhere dense;
(2) $\mathscr{C}$ is stable;
(3) $\mathscr{C}$ is monadically stable;
(4) $\mathscr{C}$ is NIP*;*
(5) $\mathscr{C}$ is monadically NIP*.*

For general hereditary classes of graphs, we do not have the collapse of the notions of stability, monadic stability, NIP, and monadic NIP stated in Theorem 1.3 for monotone classes. However, we still have the following collapses:

Theorem 1.4 ([5]). *A hereditary class of graphs is monadically* NIP *if and only if it is* NIP*. A hereditary class of graphs is monadically stable if and only if it is stable.*

The study of monadic stability and monadic NIP and their relations with first-order transductions [3] opened the way to the study of *structurally sparse* classes of graphs, that is of classes of graphs that are first-order transductions of classes of sparse graphs [6,9,10,18–20]. Intuitively, a (first-order) transduction is a way to construct a set of target graphs from the vertex-colorings of a source graph by fixed first-order formulas, and, by extension, a new class of graphs from a given class of graphs.

Extending Theorem 1.1, first-order transductions of bounded expansion classes have been characterized in terms of low shrubdepth colorings. Recall the following high level characterization of classes with bounded shrubdepth [11,12]: A class $\mathscr{D}$ has *bounded shrubdepth* if it is a transduction of a class of bounded depth rooted forests.

Theorem 1.5 ([10]). *A class $\mathscr{C}$ is a first-order transduction of a class with bounded expansion if and only if, for every parameter p, there is an integer N_p and a class $\mathscr{D}_p$ with bounded shrubdepth, such that the vertex set of each graph $G \in \mathscr{C}$ can be partitioned into at most N_p parts, each p of them inducing a subgraph in $\mathscr{D}_p$.*

Theorem 1.5 can be seen as a generalization of Theorem 1.1 as shrubdepth is a dense analogue of treedepth. On the other hand, only one direction of Theorem 1.2 has been extended to transductions of nowhere dense classes.

Theorem 1.6 ([8]). *Let $\mathscr{C}$ be a first-order transduction of a nowhere dense class. Then, for every parameter p there is a class $\mathscr{D}_p$ with bounded shrubdepth, such that for every graph $G \in \mathscr{C}$ there is an integer $N_p(G) \in |G|^{o(1)}$, with the property that the vertex set of G can be partitioned into at most $N_p(G)$ parts, each p of them inducing a subgraph in $\mathscr{D}_p$.*

Similar decompositions, where p parts induce a subgraph with bounded rankwidth were introduced in [13], while classes having such decompositions where p parts induce a subgraph with bounded linear rankwidth were discussed in [20]. However, it was not known whether such classes are monadically NIP. This question, which appears for instance in [20, Figure 3] and again in [19], will get a positive answer as a direct consequence of Theorem 2.1, which is our first main result.

The theoretical significance of first-order transductions of nowhere dense classes is witnessed by the following conjecture.

Conjecture 1.7 ([9]). *A class of graphs is monadically stable if and only if it is a first-order transduction of a nowhere dense class of graphs.*

Conjecture 1.7 can be refined as follows.

Conjecture 1.8. *For a hereditary class of graphs $\mathscr{C}$, the following properties are equivalent:*

(1) $\mathscr{C}$ is a first-order transduction of a nowhere dense class;
(2) $\mathscr{C}$ admits low shrubdepth decompositions with $n^{o(1)}$ parts;
(3) $\mathscr{C}$ is monadically stable;
(4) $\mathscr{C}$ is stable.

By Theorem 1.6, property (1) implies property (2). That property (2) implies property (3) will follow from our main result (Theorem 2.1). By Theorem 1.4, properties (3) and (4) are equivalent. Closing the chain of implications corresponds to Conjecture 1.7, which we now can decompose into two weaker statements: that property (3) implies property (2), and that property (2) implies property (1). Our second main result (Theorem 2.2) is that (3) implies (2).

2 Statement of the results

Our first main result show that NIP and stability are fixed under taking decompositions as in Theorems 1.1, 1.2, 1.5 and 1.6.

Theorem 2.1. *Let $\mathscr{C}$ be a hereditary graph class. Suppose that for every parameter p there is an* NIP *(resp. stable) class $\mathscr{D}_p$ such that for every graph $G \in \mathscr{C}$ there is an integer $N_p(G) \in |G|^{o(1)}$, with the property that the vertex set of G can be partitioned into at most $N_p(G)$ parts, each p of them inducing a subgraph in $\mathscr{D}_p$. Then $\mathscr{C}$ is* NIP *(resp. stable).*

In particular, this proves that property (2) implies property (4) in Conjecture 1.8, and so it follows that Conjectures 1.7 and 1.8 are equivalent. As mentioned after Theorem 1.6, this also proves that classes admitting low (linear) rankwidth decompositions are monadically NIP.

To place this theorem in a broader context, we introduce the notion of decomposition horizons. These seem to be of significant independent interest, and we prove some general properties. Theorem 2.1 can then be stated as "NIP and stability are decomposition horizons".

We define a *hereditary class property* to be a downset Π of hereditary graph classes, that is, a set of hereditary classes such that if $\mathscr{C} \in \Pi$ and $\mathscr{D}$ is a hereditary class with $\mathscr{D} \subseteq \mathscr{C}$, then $\mathscr{D} \in \Pi$.

Definition 1. *Let Π be a hereditary class property, let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function and let p be a positive integer. We say that a class $\mathscr{C}$ has an f-bounded* Π-decomposition *with parameter p if there exists $\mathscr{D}_p \in \Pi$ such that, for every graph $G \in \mathscr{C}$, there exists an integer $N \leq f(|G|)$ and a partition $V_1, \dots, V_N$ of the vertex set of G with $G[V_{i_1} \cup \dots \cup V_{i_p}] \in \mathscr{D}_p$ for all $i_1, \dots, i_p \in [N]$.*

When f is a constant function, we say that $\mathscr{C}$ has a *bounded-size* Π-*decomposition* with parameter p; when f is a function with $f(n) = n^{o(1)}$, we say that $\mathscr{C}$ has a *quasi-bounded-size* Π-*decomposition* with parameter p. If a class $\mathscr{C}$ has a bounded-size (resp. a quasi-bounded-size) Π-decomposition with parameter p for each positive integer p, we say that $\mathscr{C}$ has *bounded-size* Π-*decompositions* (resp. *quasi-bounded-size* Π-*decompositions*).

For instance, by Theorem 1.1 and Theorem 1.2, considering the hereditary class property "bounded treedepth", we have that a class $\mathscr{C}$ has bounded-size bounded treedepth decompositions if and only if it has bounded expansion, and it has quasi-bounded-size bounded treedepth decompositions if and only if it is nowhere dense. With these definition in hand, it is natural to consider the following constructions of graph class properties:

Definition 2. *For a hereditary class property Π we define the properties Π^+ (resp. Π^*) as follows:*

- *$\mathscr{C} \in \Pi^+$ if $\mathscr{C}$ has bounded-size Π-decompositions;*
- *$\mathscr{C} \in \Pi^*$ if $\mathscr{C}$ has quasi-bounded-size Π-decompositions.*

For every hereditary class property Π, we show that $(\Pi^+)^+ = \Pi^+$ and $(\Pi^*)^+ = \Pi^*$ (but we are not aware of any hereditary (NIP) class property Π, such that $\Pi^* \neq (\Pi^*)^*$). Also, for every two hereditary class properties Π_1 and Π_2, we show that $(\Pi_1 \cap \Pi_2)^+ = \Pi_1^+ \cap \Pi_2^+$ and $(\Pi_1 \cap \Pi_2)^* = \Pi_1^* \cap \Pi_2^*$, which suggests that, for every hereditary class property Π, there might exist an inclusion-minimum class Λ with $\Lambda^+ = \Pi^+$. On the other hand, if $(\Pi_i)_{i \in I}$ is a family of hereditary class properties indexed by a set I, then $(\bigcup_{i \in I} \Pi_i)^+ = \bigcup_{i \in I} \Pi_i^+$ and $(\bigcup_{i \in I} \Pi_i)^* = \bigcup_{i \in I} \Pi_i^*$. In particular, the inclusion order of decomposition horizons is a distributive lattice.

Definition 3. *We say that a hereditary class property Π is a* decomposition horizon *if $\Pi^* = \Pi$. If Λ is a hereditary class property, the* decomposition horizon *of Λ is the smallest decomposition horizon including Λ.*

For example, the hereditary class property of all hereditary classes excluding a fixed graph H is a decomposition horizon. We show that several hereditary class properties are decomposition horizons, including

- the class properties "bounded maximum degree after deletion of at most k vertices",
- the class property "transduction of a class with bounded maximum degree" (this property is equivalent to the model-theoretic property "mutually algebraic" [6], hence to the model-theoretic property "monadic NFCP" [14]),

- the class property "weakly sparse" (i.e. "biclique-free") of classes excluding a fixed biclique as a subgraph,
- the class property "nowhere dense".

Our examples include an infinite countable chain of decomposition horizons (the class properties"bounded maximum degree after deletion of at most k vertices"), witnessing some richness of the inclusion order on decomposition horizons.

Our second main result confirms (3) implies (2) from Conjecture 1.8.

Theorem 2.2. *Monadic stability is the decomposition horizon of the class property "bounded shrubdepth".*

From this, we obtain some combinatorial consequences for monadically stable graph classes. For example, we get the following very strong version of the Erdös-Hajnal property.

Corollary 1. *Every graph G in a hereditary stable class $\mathscr{C}$ has a clique or an independent set of size $\Omega_{\mathscr{C},\epsilon}(|G|^{1/2-\epsilon})$ for every $\epsilon > 0$. Furthermore, this cannot be improved to $\Omega_{\mathscr{C}}(|G|^{1/2})$.*

While Theorem 2.2 provides an analogue of Theorem 1.2 for monadically stable classes, monadically NIP hereditary classes seem to be more elusive. It was proved in [4] that for hereditary classes of ordered graphs, being NIP is equivalent to having bounded twin-width. On the other hand, classes with quasi-bounded-size bounded twin-width decompositions are NIP (as classes with bounded twin-width are NIP) and include transductions of nowhere dense classes (thus, conjecturally, all stable hereditary classes). Hence, it is a natural question whether every NIP hereditary class has quasi-bounded-size bounded twin-width decompositions.

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