Borel sets and reverse mathematics

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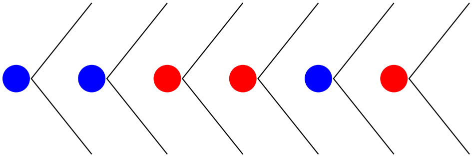
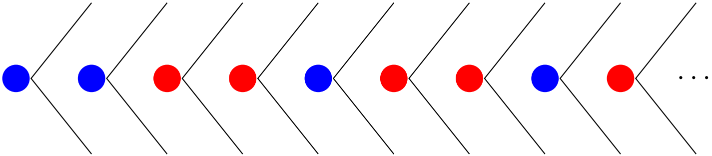
Prisoner hat game

- 6 prisoners in a line, facing the same way
- Each wears a red or blue hat
- Each sees the hats ahead of their own
- No one sees their own hat or previous hats
- Starting at the back, each tries to guess their own hat color
- They win if they make at most one mistake
- They are allowed to agree on a strategy beforehand
- Can they win?



Infinite prisoner hat game

- Same game, but now the line is infinite
- The prisoners can still win!
- But they use the Axiom of Choice (AC) to concoct their winning strategy.

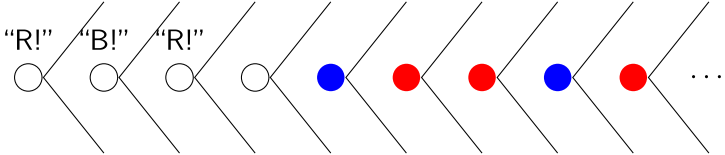


Strategies

A **strategy** is a subset $\mathcal{R} \subseteq \{R, B\}^{<\omega} \times \{R, B\}^{\omega}$, where

- $\{R, B\}^{<\omega}$ are finite strings of guesses a prisoner could hear
- $\{R, B\}^{\omega}$ are infinite strings of hat combos a prisoner could see ahead

To follow the strategy, each prisoner hearing $\sigma \in \{R, B\}^{<\omega}$ and seeing $S \in \{R, B\}^{\omega}$ will guess R if $(\sigma, S) \in \mathcal{R}$ and guess B otherwise.



$\sigma = RBR$ $S = BRRBR \cdots$

Borel strategies

A **basic open strategy** is a set of the form

$$\mathcal{U}_{\sigma,\tau} = \{(\sigma, S) : \tau \prec S\}$$

where $\sigma, \tau \in \{R, B\}^{<\omega}$.

The **Borel strategies** are defined inductively:

- Basic open strategies and their complements are Borel
- A countable intersection of Borel strategies is Borel
- A countable union of Borel strategies is Borel

Intuitively, a Borel strategy has an explicit definition. For example,

$$\mathcal{R} = \bigcup_{\sigma} \mathcal{U}_{\sigma R, \lambda}$$

is the strategy "guess R iff the prisoner behind you guessed R"

No Borel winning strategy - using measure

Prop. (Folklore) There is no Borel winning strategy for the prisoners.

Proof.

- Suppose $\mathcal{R}$ is a Borel winning strategy.
- Then $\mathcal{R}$ is measurable.
- $\mathcal{R}$ measurable $\implies$
  there is $\tau \in \{R, B\}^{<\omega}$ such that prisoner 0 says R for 99% of $S \succ \tau$.
- Then for some S', prisoner 0 says R on both $\tau RS'$ and $\tau BS'$.
- Let $S_R = B\tau RS'$ and $S_B = B\tau BS'$
- Prisoner 0 is wrong if the complete sequence of hats is S_R or S_B.
- So prisoners 1 through $|\tau|$ must guess correctly on S_R and S_B.
- Prisoner $|\tau| + 1$ hears $B\tau$ and sees S' in both cases.
- So prisoner $|\tau| + 1$ is wrong on one of S_R or S_B.

Property of Baire

Let X be a topological space (e.g. $X = \{R, B\}^{<\omega} \times \{R, B\}^{\omega}$)

- A set $A \subseteq X$ is *co-meager* if there is a sequence of dense open sets D_n such that
$$A \supseteq \bigcap_n D_n$$
- A set $B \subseteq X$ has the *property of Baire* if there is an open set $U \subseteq X$ such that U and B agree on a co-meager set.
- Call such U an *open approximation* to B.

Fact: Every Borel set has the property of Baire.

Therefore, if B is Borel and U is an open approximation to B, B is co-meager when considered as a subset of U.

No Borel winning strategy - using Baire category

Prop. (Folklore) There is no Borel winning strategy for the prisoners.

Proof.

- Suppose $\mathcal{R}$ is a Borel winning strategy.
- Then $\mathcal{R}$ **has the Baire property**.
- $\mathcal{R}$ **has the Baire property** $\implies$ there is $\tau \in \{R, B\}^{<\omega}$ such that prisoner 0 says R for **co-meager many** of $S \succ \tau$.
- Then for some S', prisoner 0 says R on both $\tau R S'$ and $\tau B S'$.
- Let $S_R = B\tau R S'$ and $S_B = B\tau B S'$
- Prisoner 0 is wrong if the complete sequence of hats is S_R or S_B.
- So prisoners 1 through $|\tau|$ must guess correctly on S_R and S_B.
- Prisoner $|\tau| + 1$ hears $B\tau$ and sees S' in both cases.
- So prisoner $|\tau| + 1$ is wrong on one of S_R or S_B.

A question

A question:

Are these different proofs?

One perspective that has been taken is:

What set existence axioms do the proofs use?

Graph coloring

Classically (using the Axiom of choice), every acyclic graph has a 2-coloring.

Theorem. (Marks '16) For every $n \geq 2$ there is a Borel n-regular acyclic graph with no Borel n-coloring.

Proof. Uses Borel Determinacy. (Much stronger than either measure or category)

Question. Is Marks' theorem above provable via measure or category? (Results of Conley, Marks & Tucker-Drob '16 indicate that the usual methods of proof via measure/category do not suffice.)

Borel Dual Ramsey Theorem

Borel Dual Ramsey Theorem (Carlson & Simpson 1984) For any $k, \ell < \omega$, suppose we ℓ-color all the k-partitions of $\mathbb{N}$. If the coloring is Borel, then there is a partition p of $\mathbb{N}$ into infinitely many pieces such that any coarsening of p down to k pieces has the same color.

Theorem (Proemel & Voigt 1985) Any coloring with the property of Baire also works.

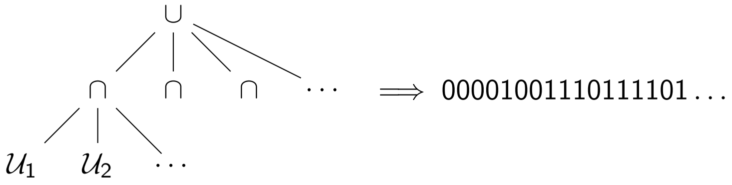
Proof is constructive.

Questions.

- Can the Borel Dual Ramsey Theorem be proved by a measure argument?
- Does the Borel Dual Ramsey Theorem imply that every Borel set has the property of Baire?

Second order arithmetic

- Most math can be carried out in second order arithmetic (SOA).
- In SOA, there are two kinds of objects, natural numbers and subsets of natural numbers (infinite bit sequences)
- Everything else is coded. For example, a Borel set B is given by a (code for a) well-founded, countably branching $\cap/\cup/$clopen-labeled tree describing how to make B.



- The axioms of SOA, including the axioms of Peano Arithmetic for the natural numbers and various set existence axioms, suffice for most mathematics outside set theory.

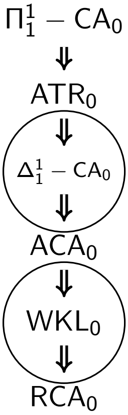
Reverse Mathematics

"When the theorem is proved from the right axioms, the axioms can be proved from the theorem." (Friedman 1968)

- Suppose Axiom A is used to prove Theorem T in SOA.
- Fix a base theory, some small fragment of axioms of SOA strong enough so T makes sense, but weaker than A.
- If T and the base theory together imply A, then A is necessary for proving T.
- The usual base theory is RCA_0, which roughly captures computable or constructive mathematics.

Axiomatic landmarks in reverse mathematics

At first, most theorems analyzed in reverse mathematics belonged to one of five equivalence classes of axiomatic power. But theorems of combinatorics proved harder to classify.



RCA_0	computable mathematics
WKL	every infinite binary tree has an infinite path
ACA_0	sets of the form $\{m \in \mathbb{N} : (\forall n \in \mathbb{N}) P(m, n)\}$ exist, P computable
ATR_0	arithmetic transfinite recursion
Π^1_1-CA	sets of the form $\{m \in \mathbb{N} : (\forall f \in \mathbb{R}) P(m, f)\}$ exist, P arithmetic

Some axioms

The strongest non-constructive theorems/axioms being used in our proofs:

- Every Borel set is measurable
- Every Borel set has the property of Baire
- Arithmetic Transfinite Recursion

We want to compare the strength of these axioms, and ask if they are needed for theorems such as

- No Borel winning strategy in the prisoner hat game
- There is a Borel 3-regular acyclic graph with no Borel 3-coloring
- The Borel Dual Ramsey Theorem
- etc.

What base theory should be used?

Borel set membership

Suppose we have a Borel set B and want to know if $X \in B$.
(B is coded by a $\cap/\cup$/clopen-labeled tree $S \subseteq \omega^{<\omega}$)

There is an inductive "procedure":

$$X \in B \iff \begin{cases} X \in B & \text{if } B \text{ is a basic open set or its complement} \\ \exists n[X \in B_n] & \text{if } B = \bigcup_n B_n \\ \forall n[X \in B_n] & \text{if } B = \bigcap_n B_n. \end{cases}$$

One step is arithmetic, and the recursion has transfinite depth.

The axiom of Arithmetic Transfinite Recursion (ATR_0) roughly states that a procedure such as the above has a well-defined output, namely an *evaluation map* $f : S \to \{0, 1\}$ which indicates X's membership status in all subtrees of S.

ATR_0 unsuitable as base theory

Over RCA_0,

- (DSFW '21) ATR_0 is equivalent to the statement that for every well-founded $\cap/\cup/$clopen-labelled tree S, there is an X which has an evaluation map in S.
- ATR_0 proves that every Borel set is measurable
- ATR_0 proves every Borel set has the property of Baire.

If ATR_0 is the base, our axioms cannot be distinguished.

Borel sets without ATR_0

Definition. (ADMSW '20) A Borel set coded by S is *completely determined (c.d.)* if every $X \in 2^\omega$ has an evaluation map in S.

Definition

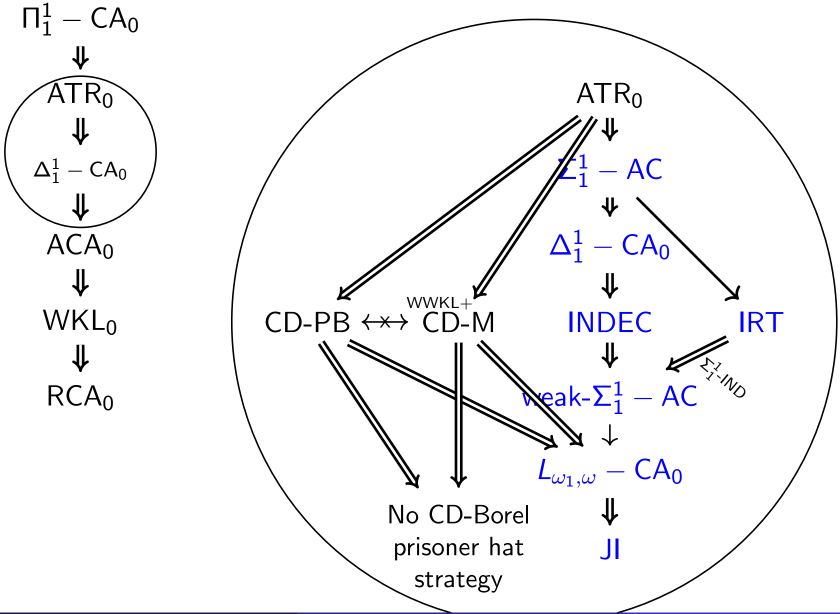
- Let CD-PB stand for "Every c.d. Borel set has the property of Baire."
- Let CD-M stand for "Every c.d. Borel set is measurable."

These statements are now well-defined *without* assuming ATR_0.

Theorem. (ADMSW '20, W '21) Both CD-PB and CD-M are strictly weaker than ATR_0.

Fact. Neither CD-PB nor CD-M implies the other. Thus, our two proofs of "no c.d.-Borel prisoner hat strategy" use different set-existence axioms.

Principles slightly weaker than ATR_0



Separations

Suppose A is an axiom (e.g. ATR_0) and T is a theorem (e.g. CD-PB).

How to show T does NOT imply A?

- Showing T does not imply A requires a *separation* - a model of SOA in which T holds but A does not.
- The SOA axioms do not guarantee the first-order part to be the true natural numbers.
- An ω-model of SOA is a model in which the first-order part is the true natural numbers.
- To specify an ω-model of SOA, we just give the second-order part, a subset $\mathcal{M} \subseteq 2^\omega$ (space of infinite bit sequences).
- ω-models are simplest and often used for separations.

Some computability theory

Let $X, Y \in 2^\omega$.

- We say X is computable if there is an algorithm which, on input n, outputs the nth bit of X.
- There are only countably many algorithms.
- The Halting set is $\{n : \text{the } n\text{th algorithm halts on input } n\}$ is a non-computable set.
- We say $X \leq_T Y$ if X can be computed by an algorithm with access to a Y oracle.
- The Halting set relative to X is

$$X' = \{n : \text{using oracle } X\text{, the } n\text{th algorithm halts on input } n\}$$

- Fact: For all X, we have $X <_T X'$

The simplest ω-model where c.d. Borel sets behave

The simpliest ω-model where c.d. Borel sets behave well is the set

$$M = HYP \subseteq 2^\omega$$

By definition:

- We say X is *hyperarithmetic* if X can be computed using any (computable) ordinal number of iterations of the halting set.
- We define

$$HYP = \{X \in 2^\omega : X \text{ is hyperarithmetic.}\}$$

Randomness or genericity at a point

A subset of 2^ω is called Δ^1_1 if it is a Borel set with a computable code.

$X \in 2^\omega$ is Δ^1_1-random if it is contained in every Δ^1_1 set of measure 1.

$X \in 2^\omega$ is called Δ^1_1-generic if it is contained in every co-meager Δ^1_1 set.

Showing measure and category weaker than ATR_0

The ω-model M which shows that CD-PB is strictly weaker than ATR_0 is

- Start with HYP
- Add some X that is Σ^1_1-generic relative to everything in M so far.
- Add everything that is hyperarithmetic relative to X.
- Repeat.

One can show this model fails ATR_0. But a c.d. Borel set B still has the property of Baire because we can "poll" all the pointwise generics about their B-membership to build up a picture of the open set U that approximates B.

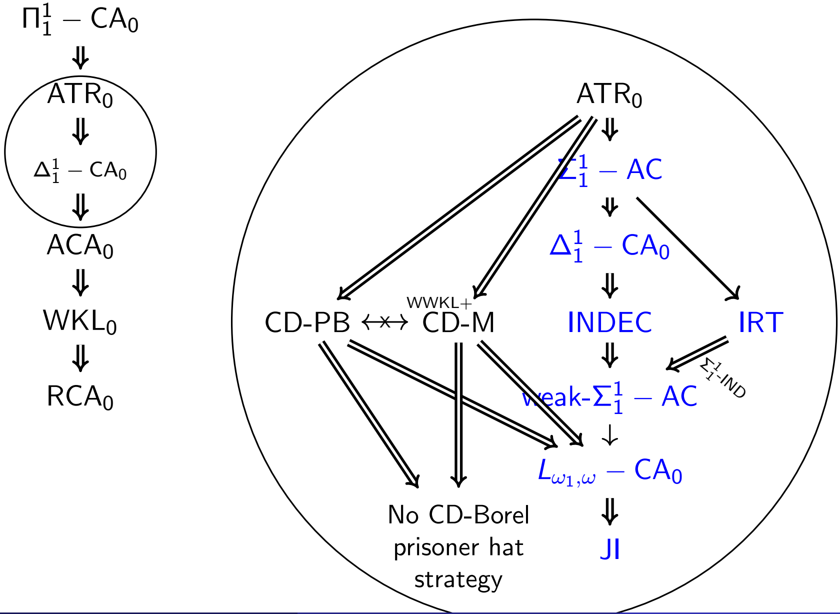
A similar idea works for CD-M.

Necessity of pointwise generic and random elements

Theorem. (ADMSW '20) For every ω-model M of CD-PB, and every $X \in M$, M also contains a Δ^1_1-generic relative to X.

Theorem. (W '21, ADMSW '20) For every ω-model M of CD-M, and every $X \in M$, M also contains a Δ^1_1-random relative to X.

Principles slightly weaker than ATR_0



A lower base theory for Borel sets

Consider $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}_0$ as a potential base theory.

- Every ω-model of $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}_0$ contains *HYP* as a subset.
- *HYP* is itself a model of $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}_0$.

Questions.

- Does "no c.d. Borel winning strategy for the prisoner hat game" follow from $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}_0$?
- Does it hold in *HYP*?
- What are c.d. Borel sets like in *HYP*?

Theorem. (Towsner, Weisshaar, W.) For any $A \subseteq HYP$, TFAE.

- There is a completely determined Borel code for A in *HYP*.
- A is $\Delta_1(L_{\omega_1^{ck}})$

$L_{\omega_1,\omega}$-CA_0 is suitable as a base theory

Corollaries (TWW). In *HYP*,

- There is a Borel well-ordering of the reals.
- The prisoners have a Borel winning strategy in the prisoner hat game.
- Let $n \in \mathbb{N}$. Every Borel n-regular acyclic graph has a Borel 2-coloring.
- The Borel Dual Ramsey Theorem fails.

Therefore, $L_{\omega_1,\omega}$-CA_0 does not imply any of the following theorems:

- No Borel winning strategy for the prisoners in the hat game.
- There is a Borel 3-regular acyclic graph with no Borel 3-coloring
- The Borel Dual Ramsey Theorem.

"Choice" arguments in *HYP*

- Using the Axiom of Choice, the prisoners do have a (very non-Borel) winning strategy in the prisoner hat game.
- Using the Axiom of Choice, every acyclic graph has a 2-coloring.
- Using the Axiom of Choice, there is a (very non-Borel) coloring of the k-partitions of $\mathbb{N}$ such that the conclusion of the Borel Dual Ramsey Theorem fails.

Strangely, the same facts are true in the Borel setting in *HYP*...

... via "the same" proofs.

However, the analogy is not perfect:

Theorem (TWW) In *HYP*, there is a Borel graph such that every vertex has degree at most 2, but this graph has no Borel 2-coloring.

Future Directions

- Is there a Borel combinatorial zoo below ATR_0? Details?
- Are there any theorems of ordinary math or Borel combinatorics equivalent to CD-PB or CD-M?
- Is there another regularity property of Borel sets which suffices to ensure those theorems about Borel sets which hold by either measure or category arguments?
- What is the reverse math strength of "There is a Borel d-regular acyclic graph with no Borel d-coloring" for $d \geq 3$? (Right now, the only known proof uses Borel Determinacy, which is not even provable in SOA.)

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