Total variants of Solovay reducibility and speedability

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Left-c.e. reals

- We start by reviewing the notion of **left computably enumerable** (or, shorty, "**left-c.e.**") **reals**

Definition

A **left-c.e. approximation** of a real α is a strictly increasing computable sequence $(a_n)_{n\in\omega}$ of dyadic rationals such that $\lim\limits_{n\to\infty} a_n = \alpha$.

A real α is **left-c.e.** if there is a left-c.e. approximation of α.

The left-c.e. reals can be equivalently defined as holding probabilities of prefix-free Turing machines (Levin, 1971, and, independently, Kraft and Chaitin, 1975).

Notion of randomness: Idea

- According to the measure-theoretic paradigm of randomness, a sequence is supposed to be random if it does not nave any "rare properties" (Downey, Hirschfeldt). Martin-Löf proposed (1966) to consider as such a rareness the belonging to some effectively null set and defined an abstract notion of a performable test for randomness, that requires from a sequence pretending to be random to pass all such tests. The resulting notion is called **Martin-Löf randomness**.

This property is equivalent to incompressibility (computational paradigm) and the inpredictability of a sequence using a computably enumerable martingale as a betting strategy (Schnorr, 1971).

- Weakening the unpredictability requirement for a sequence to be random by avoiding only computable martingale yields the notion of **Schnorr randomness**, which is equivalent to not belonging to all effectively null sets with a computable measure.

Notions of randomness: formal definitions

- A **Martin-Löf test** is a uniformly effective sequence of open sets $U_0, U_1, \ldots$, such that the set U_n has uniform measure of at most 2^{-n-1}.
- A **Schnorr test** is a Martin-Löf if the measures of U_n are uniformly computable.
- A **Solovay test** is a uniformly effective sequence of intervals $I_0, I_1, \ldots$, such that the sum of uniform measures of all I_n is finite.
- A **total Solovay test** is a Solovay test which has a computable measure.

Definition

A sequence A is **Martin-Löf random**, if there is no Martin-Löf test $U_0, U_1, \ldots$, such that $A \in \bigcap_{i \in \omega} U_i$.

A sequence A is **Schnorr random**, if there is no Schnorr test $U_0, U_1, \ldots$, such that $A \in \bigcap_{i \in \omega} U_i$.

- It is well-known that a sequence A is Martin-Löf random if and only if there is no Solovay test $I_0, I_1, \ldots$ such that A is contained in infinitely many of the sets I_n as well as Schnorr random if and only if there is no total Solovay test $I_0, I_1, \ldots$ such that A is contained in infinitely many of the sets I_n

Further, we identify any real $\alpha := 0.A$ with its binary representation A.

Notions of randomness: an example

The Martin-Löf random are reals whose binary representations don't belong to some effective null set.

- A classical (Chaitin, 1975) example of a Martin-Löf random left-c.e. reals is a **Chaitin's Omega number** Ω, which is equal to the halting probability of some **optimal** prefix-free machine, i.e., a prefix-free machine designed to simulate every other prefix-free machine.

Notion of reducibility

By reducibility we mean a transitive and reflexive preorder on the Cantor space designed to witness that one sequence is in some sense more "complex" than another one.

- The most natural way to compare the complexity of sequences is the comparison of their initial segments, the most common example of this approach is the Turing reducibility $\leq_T$.

Example

A sequence A is K-reducible to a sequence B if the prefix-free Kolmogorov complexity of the initial segments of A is (up to an additive constant) not higher than that of B:

$$A \leq_K \beta :\Longleftrightarrow K(A \upharpoonright n) \leq K(B \upharpoonright n) + O(1).$$

- The straightforward application of the concept of speedability on $\mathbb{R}$ can be reached by identifying the real numbers with their (binary) representations.
- Another possibility to define a "real" reducibility is to consider a translation function from an approximation of a real to an approximation of another one with not worse accuracy.

Solovay reducibility: idea

The main idea of a reducibility of a left-c.e. reals to another that preserves the Martin-Löf nonrandomness is motivated by the following observation:

- given two left-c.e. reals α and β and a computable translation function f from a rational numbers left from β to rational numbers left from α lying at least (up to a multiplicative constant) so close to α as the initial number to β, we can compute a Solovay test containing α from every Solovay test containing β in a very simple way.

Solovay reducibility

Definition

A real α is **Solovay reducible** to a real β, written $\alpha \leq_S \beta$, if there is a constant $c > 0$ and a partial computable function $\varphi\colon \mathbb{Q} \to \mathbb{Q}$ such that for all $q < \beta$ it holds that $\varphi(q)\downarrow < \alpha$ and $\alpha - \varphi(q) < c(\beta - q)$.

The degree structure induced by $\leq_S$ on the set of left-c.e. reals is an uppersemilattice with the following properties:

- the least upper bound of α and β is given by $\alpha + \beta$,
- least degree: computable reals,
- greatest degree: left-c.e. Martin-Löf random reals (see next slide for details).

Outside of left-c.e. reals (to whom it may concern):

- $\leq_S$ implies the both-sided extention of the Solovay reducibility with an additional term $\leq_S^{1a}$ introduced by Zheng and Rettinger (2004) and coinside with them on left-c.e. reals.

The greatest degree of the Solovay uppersemilattice

- From one hand, $\alpha \leq_S \Omega$ for every left-c.e. number α since the optimal prefix-free machine can simulate every prefix-free machine within the same time up to a constant.
- From other hand, every two Martin-Löf random left-c.e. Ω^1 and Ω^2 lie in the same Solovay-degree (Kučera, Slaman, 2001)
- ...and even have a limit of the sequence $(\frac{\Omega^1-\omega_n^1}{\Omega^2-\omega_n^2})_{n\to\infty}$ which does not depend on the choice of left-c.e. approximations $\omega_n^1 \nearrow \Omega^1$ and $\omega_n^2 \nearrow \Omega^2$ (Barmpalias, Lewis-Pye, 2016).

Setting $\Omega_2 := \Omega_1$ in the last fraction implies that a left approximation of a left-c.e. Martin-Löf random numbers can never be "speeded up" using a computable index function. The corresponding notion of speedability has been formally defined by Merkle and Titov (2020), see the next slide.

Speedability of left-c.e. reals

Definition

A computable nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ which fulfills $f(n) \geq n$ for every n is called an **index speed-up function**.

A left-c.e. real α is **ρ-speedable with respect to its left-c.e. approximation** $(a_n)_{n\to\infty}$ for a constant $\rho \in (0, 1)$ if there is an index speed-up function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\liminf_{n\to\infty} \frac{\alpha - a_{f(n)}}{\alpha - a_n} \leq \rho.$$

A left-c.e. real α is **ρ-speedable** if it is ρ-speedable with respect to some of its left-c.e. approximations.

- Speedability is a Solovay degree property and does not depend on the choice of $\rho \in (0, 1)$ (Merkle, Titov, 2020).
- Martin-Löf random reals left-c.e. reals are nonspeedable.
- It is still unknown, whether there exist any nonspeedable Martin-Löf nonrandom left-c.e. real.

Schnorr reducibility

Definition

A real α is **Schnorr reducible** to a real β, written $\alpha \leq_{\mathrm{Sch}} \beta$, if there is a constant $c > 0$ such that for every computable measure machine B there exists a computable measure machine A so that

$$\mathrm{K}_{\mathrm{A}}(\alpha \upharpoonright n) \leq \mathrm{K}_{\mathrm{B}}(\beta \upharpoonright n) + c$$

A real α is **uniform Schnorr reducible** to a real β, written $\alpha \leq_{\mathrm{uSch}} \beta$, if there is a uniform way to transform B in A.

- Least degree: computable reals.
- Greatest degree: Schnorr random reals.
- There is a pair α, β of left-c.e. reals such that $\alpha \not\leq_{\mathrm{Sch}} \alpha + \beta$, so addition is not an upper bound for the Schnorr randomness (Miyabe, Nies and Stephan 2018).

Motivation

The Schnorr random reals are not closed upwards in the Solovay degrees. (Miyabe, Nies and Stephan 2018)

Idea (Downey, 2018)

Requiring the translation function of a Solovay reducibility can provide the way to transform a total Solovay test into a total Solovay test.

Total Solovay reducibility

Definition

A real α is total Solovay reducible to a real β, written $\alpha \leq_{\mathrm{S}}^{\mathrm{tot}} \beta$, if there is a constant $c > 0$ and a total computable function $f\colon \mathbb{Q} \to \mathbb{Q}$ such that for all $q < \beta$ it holds that $f(q) < \alpha$ and $\alpha - f(q) < c(\beta - q)$.

- The total Solovay reducibility is a reflexive and transitive preorder that implies the normal Solovay reducibility as well as the uniform Schnorr reducibility.

The previous fact implies the following straightforward corollaries:

- Martin-Löf random reals are closed upwards in the $\leq_{\mathrm{S}}^{\mathrm{tot}}$-degrees of left-c.e. reals.
- Schnorr random reals are closed upwards in the $\leq_{\mathrm{S}}^{\mathrm{tot}}$-degrees of left-c.e. reals.

Total Solovay reducibility

In contrast to the Solovay uppersemilattice, the computable reals don't form a least total Solovay degree, the easiest counterexample is a hyperimmune left-c.e. real to which the computable real are not $\leq_S^{tot}$-reducible. More over, we can construct an antichain of such left-c.e. reals, so that each one of them will be hyperimmune over the previous one. The latter observation implies the following fact:

- There exists a countably infinite antichain of $\leq_{S^2}$-incomparable left-c.e. reals, so that all of them are incomparable with the computable reals.
- Therefore, the least total Solovay degree does not exist.

Zheng and Rettinger have shown (2004) that putting an additional term into a Solovay reducibility condition can fix the similar deficiency of a Solovay reducibility outside of left-c.e. reals. A similar approach can be used for the total Solovay degrees to provide the reducibility of a computable real to every left-c.e. real.

Total Solovay reducibility with an additional term

Definition

A real α is **total additive reducible** to a real β, written $\alpha \leq_{\mathrm{S}}^{\mathrm{tot}a} \beta$, if there is a constant c and a total computable function $f : \mathbb{Q} \to \mathbb{Q}$, such that if $q \in \mathbb{Q}$ and $q < \beta$, then $f(q) < \alpha$ and $\alpha - f(q) < c(\beta - q + 2^{-|q|})$.

- As well as the non-additive one, the additive total Solovay reducibility a reflexive and transitive preorder that implies both normal Solovay reducibility and the uniform Schnorr reducibility.
- Martin-Löf random reals and Schnorr random reals are still closed upwards in the $\leq_{\mathrm{S}}^{\mathrm{tot}a}$-degrees of left-c.e. reals.
- The computable reals form the least additive Solovay degree.

Speedability: a functional approach

Definition

A computable nondecreasing function $g :\subseteq \mathbb{Q}_2|_{[0,1)} \to \mathbb{Q}_2|_{[0,1)}$ which is defined on an interval I and fulfills $f(q) \geq q$ on this interval is called a **speed-up function** on I.

A real α is **ρ-speedable** for a constant $\rho \in (0,1)$ if there is a speed-up function g on $[0,\alpha)$ and a sequence $q_n \nearrow \alpha$ of dyadic rationals converging to α such that

$$\liminf_{n\to\infty} \frac{\alpha - g(q_n)}{\alpha - q_n} \leq \rho.$$

- On the set of left-c.e. reals, the new generalized definition of ρ-speedability is equivalent to a standard one.
- Merkle and Titov showed (2021) that the nonspeedability of Martin-Löf random reals holds true even outside of left-c.e. real.

A total version of speedability

Definition

A real α is **total ρ-speedable** for a constant $\rho \in (0,1)$ if there is a total speed-up function g on $[0,\alpha)$ and a sequence $q_n \nearrow \alpha$ of dyadic rationals converging to α such that

$$\liminf_{n\to\infty} \frac{\alpha - g(q_n)}{\alpha - q_n} \leq \rho.$$

- Similar to the ordinary one, the total speedability is a total Solovay degree property and does not depend on the choice of $\rho \in (0,1)$.
- Obviously, the Martin-Löf random reals are never total speedable.

Speed-up functions and sets of limit points

Definition

For a real α, a function $g : \mathbb{Q}_2|_{[0,\alpha)} \to \mathbb{Q}_2|_{[0,\alpha)}$ is called **almost accumulation point-free (aapf)** if it holds that

$$\lambda\Big(\bigcup_{q \in [0,\alpha)} [q, g(q)]\Big) = \alpha,$$

where λ means the Lebesgue measure.

- Notice that, for a left-c.e. real α, the speedability automatically implies the speedability via an aapf function on $[0, \alpha)$.

Theorem

For every function $g : \mathbb{Q}_2|_{[0,\alpha)} \to \mathbb{Q}_2|_{[0,\alpha)}$ and every constant $c \in [0, a - b]$, the following statements are equivalent:

- *g is aapf on $[0, \alpha)$,*
- $\mu\{x \in [0, \alpha) : x \notin \bigcup_{q \in [0,\alpha)} [q, g(q)]\} = \alpha$
- $\mu\{x \in [0, \alpha) : \exists q_n \nearrow x\text{:}\ g(q_n) \nearrow x\} = \alpha$
- *For the partial function $g^*(x) := \lim_{q \nearrow x} g(q)$ from $[0, \alpha]$ onto itself, one holds:*
  $\lambda\{x \in [0, \alpha) : g^*(x) \text{ is defined and } g^*(x) = x\} = \alpha.$

Total speedability and Schnorr-random reals

Theorem

The Schnorr random left-c.e. reals are never speedable via an aapf function on $[0, 1)$.

- Similar to the nonpeedability of Martin-Löf reals, this result can also be extended outside of left-c.e. reals.

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