The Foundations of Arithmetic: Peano vs Dedekind

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Introduction

1889: *Arithmetices principia, nova methodo exposita* (AP) by Giuseppe Peano

1888: *Was sind und was sollen die Zahlen?* (WSZ) by Richard Dedekind.

Some writers talk about Dedekind-Peano's axioms instead of Peano's axioms. How much did Dedekind's work affect Peano's work? What are the similarities and differences between the two texts?

Peano himself, in the introduction to his work states that the work of Dedekind was very useful.

Dedekind had a public debate on the foundation of Arithmetic for at least five years from 1872 to 1878.

Peano studied at the University of Turin from 1876 to1880.

According to Volterra in the second half of the 19th century, there were contacts between Italian and German universities.

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Method and Objective

PEANO

In his introduction to the (AP), Peano: the difficulty of foundations is due to the ambiguity of language and that it is very important to examine carefully every word we use.

The goal of his work: the presentation of a method that emerged after the examination of the problem as well as an application on Arithmetic.

His book was an introduction to logical symbolism.

His method is nothing more than the use of a symbolic language.

DEDEKIND

Shows particular interest in the study of the axiomatic properties of numbers as well as in isolating the properties from their numerical character so that they can be incorporated into more general concepts.

Begins with informal references to some basic principles of set theory, beginning with the definition of the important concept of chain.

Definition: K is called a *chain* when $K' \ni K$.
(Let φ be a mapping of the system K. Then $K' = \varphi(K)$).

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The number, the axioms and the propositions

PEANO

Didn't give definitions for the number, the unit, and successor.

DEDEKIND

Gave a definition using the concept of chain:

«<u>Definition:</u> A system N is said to be *simply infinite* when there exists a similar mapping φ of N into itself such that N appears as the chain (44) of an element not contained in $\varphi(N)$. We call this element, which we shall denote in what follows by the symbol 1, the *base-element* of N, and say that the simply infinite system N is *ordered* by this mapping φ. If we retain the earlier convenient symbols for images and chains (§4) then the essence of a simply infinite system N consists in the existence of a mapping φ of N and an element 1 which satisfy the following conditions α, β, γ, δ:

α. $N' \ni N$.

β. $N = 1_0$.

γ. The element 1 is not contained in N'.

δ. The mapping φ is similar.

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The number, the axioms and the propositions

PEANO

Didn't give definitions for the number, the unit, and successor.

DEDEKIND

Gave a definition using the concept of chain:

«Definition: If in the consideration of a simply infinite system N ordered by a mapping φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the ordering mapping φ, then these elements are called *natural numbers* or *ordinal numbers* or *simply numbers*, and the base-element 1 is called the *base-number* of the number-series N.»

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The number, the axioms and the propositions

PEANO

The axioms given by Peano in his original text are nine of which four refer to the relation of equality.

1. 1 ε N
2. a ε N .Ɔ. a = a.
3. a, b ε N .Ɔ: a = b .=. b = a.
4. a, b, c ε N .Ɔ.·. a = b . b = c :Ɔ. a = c.
5. a = b . b ε N :Ɔ. a ε N.
6. a ε N .Ɔ. a + 1 ε N.
7. a, b ε N .Ɔ: a = b .=. a + 1 = b + 1.
8. a ε N .Ɔ. a + 1-= 1.
9. k ε K .·. 1 ε k .·. x ε N . x ε k :Ɔ$_{x}$. x + 1 ε k :: Ɔ. N Ɔ k.

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The number, the axioms and the propositions

PEANO

In *Rivista di matematica 1* he transforms his system by giving the five known axioms:

«1. 1 ε N
2. + ε N\N
3. a, b ε N . a + = b + : Ɔ . a = b
4. 1 - ε N +
5. s ε K . 1 ε s . s + Ɔ s : Ɔ . N Ɔ s»

The first four axioms of Peano are indeed identical to those which Dedekind incorporated in his definition of natural numbers.

DEDEKIND

Axioms:

α. $N' \ni N$.
β. $N = 1_0$.
γ. The element 1 is not contained in N'.
δ. The mapping φ is similar.

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The number, the axioms and the propositions

PEANO

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The 5th axiom, which is nothing more than mathematical induction, is given by Peano as an axiom while Dedekind proves it as a proposition

DEDEKIND

«Theorem of complete induction (inference from *n* to *n*′): In order to show that a theorem holds for all numbers *n* in a chain m_0, it is sufficient to show,

ρ. That it holds for *n* = *m*, and
σ. that from the validity of the theorem for a number *n* of the chain m_0 its validity for the following number *n*′ always follows.

This results immediately from the more general theorems (59) or (60). The most frequently occurring case is when *m* = 1 and therefore m_0 is the complete number-series *N*.»

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The number, the axioms and the propositions

PEANO

Doesn’t define addition while all its properties are proven using induction.

Subtraction, multiplication, and powers are defined recursively.

Peano's text uses the language of logic as a tool.

AP is easier to read but does not fully prove all the properties of numbers considering that the reader can practice the method on his own by proving the sentences.

Finally, Peano's presentation is much closer to what we know today about Arithmetic.

DEDEKIND

Dedekind follows a similar path for properties and operations within natural numbers, with the main difference being the recursive definition of addition.

Dedekind instead wanted to reduce these "axioms" to deep logical principles

Dedekind's book is much more detailed but also more difficult for the reader.

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Conclusions

When *Was sind und was sollen die Zahlen?* was published, Dedekind was a famous mathematician. He worked on the foundations of Arithmetic for several years and for which he exchanged views with other mathematicians of his time.

Peano's research interest was to look for errors in the notes of Calculus, to correct them and to offer a lesson as comprehensible as possible to his students. In his view, the errors were due to natural language, so he studied a way to express the theories he studied in symbolic language, which was also Leibniz's vision.

Immediately after the *Arithmetices principia, nova methodo exposita*, Peano publishes the *I Principii Di Geometria Logicamente Esposti* in which he attempts to apply his method in Geometry as well.

In 1897, at the 1st World Mathematical Congress in Zurich, he presented his papers, and claimed to have answered Leibniz's question:

«After two centuries, this "dream" of the inventor of the infinitesimal calculus has become a reality…. We now have the solution to the problem proposed by Leibniz. I say "the solution" and not "a solution", for it is unique. Mathematical logic, the new science resulting from this research, has for its object the properties of the operations and relations of logic. Its object, then, is a set of truths, not conventions.»

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Conclusions

Peano's main goal was to apply a symbolic language to a theory

Dedekind was interested in the foundation of Arithmetic, something he had been studying and discussing for about 17 years.

Their work presents the same theory with different methods of approach.

After the publication of *Arithmetices principia, nova methodo exposita*, Peano read *Was sind und was sollen die Zahlen?*, given that he "omitted" four of his original axioms, resulting in 4 of the final 5 to being similar to those of Dedekind.

The main differences between the two texts is the definition of number.

Another difference between the two texts has to do with mathematical induction. While Peano accepts it as an axiom, Dedekind presents it as a theorem, which he proves.

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Thank you for your attention!!

Katerina Petsi