Proof-relevance in Bishop-style constructive mathematics

Iosif Petrakis

University of Munich

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Overview

- Bishop Set Theory (BST)
- A (partial) BHK interpretation of BISH in BST
- Martin-Löf sets in BST

Bishop Set Theory (BST)

Theories of Bishop sets

- Bishop's "official" theory of sets (oBST) was presented in [2, 3]. In contrast to Brouwer's INT, Bishop's informal system BISH did not contradict CLASS
- It motivated Martin-Löf's type theory (MLTT) [8, 9] and most of the formal studies of the 70's (see [1]).
- A "naive" theory of Bishop sets (nBST) was used by Bridges and Richman [4, 10].
- The type-theoretic interpretation of oBST into the theory of setoids [12, 13] is nowadays the standard way to understand Bishop sets.
- A categorical interpretation of Bishop sets is Palmgren's constructive adaptation [11] of Lawvere's elementary theory of the category of sets.
- In [5] Coquand views Bishop sets as a natural sub-presheaf of the universe in the cubical set model.
- In [17] a reconstruction (BST) of oBST is given, highlighting dependent assignment routines, predicativity, and set-indexed families of sets and subsets.

Fundamentals of BST

Basic judgments $m \in \mathbb{N}$, $m := n, \mathbb{N} \in \mathbb{V}_0$,

$\alpha \in \mathcal{O}(A, B)$ i.e., $\alpha \colon A \rightsquigarrow B$, and A, B sets,

$$\mathcal{A} \in \mathcal{DO}(I, \Lambda) \qquad \mathcal{A} \colon \bigcurlywedge_{i \in I} \lambda_0(i) \qquad \mathcal{A}_i \in \lambda_0(i), \qquad \lambda_0 \colon I \rightsquigarrow \mathbb{V}_0$$

Defined judgments $a \in A$, for a defined totality A.

An open-ended universe $\mathbb{V}_0$ of predicative sets, a proper class and univalent (its canonical equality is a version of Voevodsky's UA).

A defined Bishop set is a predicative totality A (the construction $a \in A$ does not involve quantification over $\mathbb{V}_0$), together with an equivalence relation $a =_A a'$.

The function-extensionality axiom added to MLTT is built-in as the canonical equality of $\mathcal{O}(A, B)$ or $\mathcal{DO}(I, \Lambda)$.

$A \times B, A + B, \mathbb{1}, \mathbb{2}$, no empty set.

$\mathcal{P}(A)$ is a proper class, there is only $\varnothing_A$, where A inhabited set.

Formulas

Prime formulas:

$s =_{\mathbb{N}} t$, $s \neq_{\mathbb{N}} t$, where s, t are elements of $\mathbb{N}$.

Complex formulas:

If A, B are formulas, then $A \vee B$, $A \wedge B$, $A \Rightarrow B$ are formulas.

If S is a set and $\phi(x)$ is a formula, for every variable x of set S, then $\exists_{x \in S}\big(\phi(x)\big)$ and $\forall_{x \in S}\big(\phi(x)\big)$ are formulas.

Weak and strong negation, strong implication

$$\neg A := A \Rightarrow \bot$$

$$\bot := 0 =_{\mathbb{N}} 1$$

$$\top := 0 \neq_{\mathbb{N}} 1$$

$$\neg(s =_{\mathbb{N}} t) := s \neq_{\mathbb{N}} t \ \ \& \ \ \neg(s \neq_{\mathbb{N}} t) := s =_{\mathbb{N}} t.$$

$$\neg(A \vee B) := \neg A \wedge \neg B$$

$$\neg(A \wedge B) := \neg A \vee \neg B$$

$$\neg(A \Rightarrow B) := A \wedge \neg B$$

$$\neg\left(\exists_{x \in S} \phi(x)\right) := \forall_{x \in S}\left(\neg\phi(x)\right)$$

$$\neg\left(\forall_{x \in S} \phi(x)\right) := \exists_{x \in S}\left(\neg\phi(x)\right)$$

$$A \Rightarrow B := (A \Rightarrow B) \wedge (\neg B \Rightarrow \neg A)$$

Various categories of sets

Set : category of sets and functions

$\mathbf{Set}^{\mathrm{se}}$: category of sets with inequality and s.e. functions

Set : category of strong sets and strong functions

Constructive measure theory within $\mathbf{Set}^{\mathrm{se}}$

I-families of sets

If $D(I) := \{(i,j) \in I \times I \mid i =_I j\}$, a family of sets indexed by I is a pair $\Lambda := (\lambda_0, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and

$$\lambda_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big),$$

such that, if $\lambda_1(i,j) := \lambda_{ij}$, for every $(i,j) \in D(I)$,

(a) For every $i \in I$, we have that $\lambda_{ii} = \mathrm{id}_{\lambda_0(i)}$.

(b) If $i =_I j$ and $j =_I k$, the following diagram commutes

$$\begin{array}{ccc} \lambda_0(i) & & \\ {\scriptstyle \lambda_{ij}} \downarrow & \searrow {\scriptstyle \lambda_{ik}} & \\ \lambda_0(j) & \xrightarrow[\lambda_{jk}]{} & \lambda_0(k). \end{array}$$

If $i =_I j$, we call the function λ_{ij} the transport map from $\lambda_0(i)$ to $\lambda_0(j)$, and we call λ_1 the modulus of function-likeness of λ_0:

$$(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j).$$

The Sigma-set and the Pi-set of a family

The Sigma-set $\sum_{i\in I}\lambda_0(i)$ of Λ is defined by

$$w \in \sum_{i\in I}\lambda_0(i) :\Leftrightarrow w := (i,x); \quad i \in I, x \in \lambda_0(i)$$

$$(i,x) =_{\sum_{i\in I}\lambda_0(i)} (j,y) :\Leftrightarrow i =_I j \ \& \ \lambda_{ij}(x) =_{\lambda_0(j)} y.$$

The Pi-set $\prod_{i\in I}\lambda_0(i)$ of dependent functions over Λ is defined by

$$\Phi \in \prod_{i\in I}\lambda_0(i) :\Leftrightarrow \Phi \in \mathcal{DO}(I,\Lambda) \ \& \ \forall_{(i,j)\in D(I)}\big(\Phi_j =_{\lambda_0(j)} \lambda_{ij}(\Phi_i)\big),$$

and it is equipped with the pointwise equality of $\mathcal{DO}(I,\Lambda)$.

Set-relevant families of sets

$\Lambda^* := \big(\lambda_0, \varepsilon_0^\lambda, \lambda_2\big)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, $\varepsilon_0^\lambda : D(I) \rightsquigarrow \mathbb{V}_0$, and

$$\lambda_2 : \bigcurlywedge_{(i,j)\in D(I)} \bigcurlywedge_{p\in\varepsilon_0^\lambda(i,j)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_2\big((i,j), p\big) := \lambda_{ij}^p, \quad (i,j) \in D(I),$$

(i) *For every $i \in I$ there is $p \in \varepsilon_0^\lambda(i,i)$ s.t.* $\lambda_{ii}^p =_{\mathbb{F}(\lambda_0(i),\lambda_0(i))} \mathrm{id}_{\lambda_0(i)}$.

(ii) *For every $(i,j) \in D(I)$ and $p \in \varepsilon_0^\lambda(i,j)$ there is $q \in \varepsilon_0^\lambda(j,i)$ s.t.*

$$\begin{array}{ccc} \lambda_0(i) & & \\ {\scriptstyle \lambda_{ij}^p}\downarrow & \searrow^{\mathrm{id}_{\lambda_0(i)}} & \\ \lambda_0(j) & \xrightarrow[\lambda_{ji}^q]{} & \lambda_0(i) \end{array} \qquad \begin{array}{ccc} \lambda_0(i) & & \\ {\scriptstyle \lambda_{ij}^p}\downarrow & \searrow^{\lambda_{ik}^r} & \\ \lambda_0(j) & \xrightarrow[\lambda_{jk}^q]{} & \lambda_0(k). \end{array}$$

(iii) *For every $p \in \varepsilon_0^\lambda(i,j)$ and $q \in \varepsilon_0^\lambda(j,k)$ there is $r \in \varepsilon_0^\lambda(i,k)$ s.t.*

function-like: $\forall_{(i,j)\in D(I)} \forall_{p,p'\in\varepsilon_0^\lambda(i,j)} \big(p =_{\varepsilon_0^\lambda(i,j)} p' \Rightarrow \lambda_{ij}^p = \lambda_{ij}^{p'}\big)$

The Sigma-set and the Pi-set of a set-relevant family

$\sum_{i \in I}^{*} \lambda_0(i)$ is $\sum_{i \in I} \lambda_0(i)$, equipped with the following equality

$$(i, x) =_{\sum_{i \in I}^{*} \lambda_0(i)} (j, y) :\Leftrightarrow i =_I j \ \& \ \exists_{p \in \varepsilon_0^{\lambda}(i,j)} \big(\lambda_{ij}^{p}(x) =_{\lambda_0(j)} y\big).$$

$\prod_{i \in I}^{*} \lambda_0(i)$ is defined by

$$\Theta \in \prod_{i \in I}^{*} \lambda_0(i) :\Leftrightarrow \Theta \in \mathcal{DO}(I, \lambda_0) \ \& \ \forall_{(i,j) \in D(I)} \forall_{p \in \varepsilon_0^{\lambda}(i,j)} \big(\Theta_j =_{\lambda_0(j)} \lambda_{ij}^{p}(\Theta_i)\big)$$

$$w =_{\sum_{i:\ I} P(i)} w' \simeq \sum_{p:\ \mathtt{pr}_1(w) = \mathtt{pr}_1(w')} p_*(\mathtt{pr}_2(w)) = \mathtt{pr}_2(w'),$$

If $\Phi : \prod_{i \in I} P(i)$, there is a term

$$\mathtt{apd}_{\Phi} : \prod_{p:\ i = j} \big(p_*(\Phi_i) = \Phi_j\big).$$

A (partial) BHK interpretation of BISH in BST

Membership with evidence

Let X, Y be sets, and let $P(x)$ be a property on X of the form

$$P(x) :\Leftrightarrow \exists_{p \in Y}\big(Q(x,p)\big),$$

$$\big[x =_X x' \ \& \ p =_Y p' \ \& \ Q(x,p)\big] \Rightarrow Q(x',p'),$$

Let $\texttt{PrfMemb}_0^P : X \rightsquigarrow \mathbb{V}_0$, defined by

$$\texttt{PrfMemb}_0^P(x) := \{p \in Y \mid Q(x,p)\},$$

$$\texttt{PrfMemb}_1^P : \bigcurlywedge_{(x,x') \in D(X)} \mathbb{F}\big(\texttt{PrfMemb}_0^P(x), \texttt{PrfMemb}_0^P(x')\big)$$

$$\texttt{PrfMemb}_{xx'}^P := \texttt{PrfMemb}_1^P(x,x') : \texttt{PrfMemb}_0^P(x) \to \texttt{PrfMemb}_0^P(x')$$

$$\texttt{PrfMemb}_{xx'}^P(p) := p$$

$P(x)$ extensional, $\texttt{PrfMemb}^P := \big(\texttt{PrfMemb}_0^P, \texttt{PrfMemb}_1^P\big) \in \texttt{Fam}(X)$.

Cauchy sequences: $X := \mathbb{F}(\mathbb{N}, \mathbb{R})$, $Y := \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)$, $(x_n)_{n\in\mathbb{N}} \in \mathbb{F}(\mathbb{N}, \mathbb{R})$

$$\mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big) :\Leftrightarrow \exists_{C\in\mathbb{F}(\mathbb{N}^+,\mathbb{N}^+)}\big(C\colon \mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big),$$

$$C\colon \mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big) :\Leftrightarrow \forall_{k\in\mathbb{N}^+}\forall_{n,m\geqslant C(k)}\left(|x_n - x_m| \leqslant \frac{1}{k}\right).$$

$$R\big((x_n)_{n\in\mathbb{N}}, C\big) :\Leftrightarrow \mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big) \quad \text{extensional}$$

$$\mathtt{PrfMemb}^{\mathrm{Cauchy}} := \big(\mathtt{PrfMemb}_0^{\mathrm{Cauchy}}, \mathtt{PrfMemb}_1^{\mathrm{Cauchy}}\big) \in \mathtt{Fam}\big(\mathbb{F}(\mathbb{N}, \mathbb{R})\big)$$

$$\mathtt{PrfMemb}_0^{\mathrm{Cauchy}}\big((x_n)_{n\in\mathbb{N}}\big) := \big\{C \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+) \mid C\colon \mathrm{Cauchy}\big((x_n)_{n\in\mathbb{N}}\big)\big\}$$

Partial BHK-interpretation of BISH within BST I

$$\mathtt{Prf}(\top) := \mathbb{1}$$

If $x \in X_P :\Leftrightarrow \exists_{p \in Y}\big(Q(x,p)\big)$, let

$$\mathtt{Prf}(x \in X_P) := \mathtt{PrfMemb}_0^P(x)$$

If ϕ, ψ formulas s.t. $\mathtt{Prf}(\phi), \mathtt{Prf}(\psi)$ are defined, let

$$\mathtt{Prf}(\phi \ \& \ \psi) := \mathtt{Prf}(\phi) \times \mathtt{Prf}(\psi),$$

$$\mathtt{Prf}(\phi \vee \psi) := \mathtt{Prf}(\phi) + \mathtt{Prf}(\psi),$$

$$\mathtt{Prf}(\phi \Rightarrow \psi) := \mathbb{F}\big(\mathtt{Prf}(\phi), \mathtt{Prf}(\psi)\big).$$

Partial BHK-interpretation of BISH within BST II

Let $\phi(x)$ be a formula on X, and $\mathtt{Prf}^{\phi} := \big(\mathtt{Prf}_0^{\phi}, \mathtt{Prf}_1^{\phi}\big) \in \mathtt{Fam}(X)$, where $\mathtt{Prf}_0^{\phi} \colon X \rightsquigarrow \mathbb{V}_0$ is given by the rule

$$x \mapsto \mathtt{Prf}_0^{\phi}(x) := \mathtt{Prf}\big(\phi(x)\big)$$

The $\mathtt{Prf}$-sets of the formulas $\forall_{x \in X}\phi(x)$ and $\exists_{x \in X}\phi(x)$ with respect to the given family $\mathtt{Prf}^{\phi}$ are defined by

$$\mathtt{Prf}\left(\forall_{x \in X}\phi(x)\right) := \prod_{x \in X} \mathtt{Prf}_0^{\phi}(x) := \prod_{x \in X} \mathtt{Prf}\big(\phi(x)\big),$$

$$\mathtt{Prf}\left(\exists_{x \in X}\phi(x)\right) := \sum_{x \in X} \mathtt{Prf}_0^{\phi}(x) := \sum_{x \in X} \mathtt{Prf}\big(\phi(x)\big).$$

Martin-Löf sets in BST

Totalities with a proof-relevant (existential) equality

- $\mathbb{V}_0$, $\mathcal{P}(X)$, $\texttt{Fam}(I)$
- The Richman ordinals
- The Sigma-set of a direct family of sets
- The integrable functions L^1 of a Bishop-Cheng integration space L.
- The set of reals $\mathbb{R}$:

$$x =_{\mathbb{R}} y \Leftrightarrow \exists_{\omega \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+)} \forall_{j \in \mathbb{N}^+} \forall_{n \geqslant \omega(j)} \left(|x_n - y_n| \leqslant \frac{1}{j} \right)$$

$$\mathrm{Eq}(x, y) := \{\omega \in \mathbb{F}(\mathbb{N}^+, \mathbb{N}^+) \mid \omega : x =_{\mathbb{R}} y\},$$

$$\omega : x =_{\mathbb{R}} y :\Leftrightarrow \omega \geqslant \mathrm{id}_{\mathbb{N}^+} \ \& \ \forall_{j \in \mathbb{N}^+} \forall_{n \geqslant \omega(j)} \left(|x_n - y_n| \leqslant \frac{1}{j} \right)$$

Martin-Löf set $\widehat{X} := (X, =_X, \mathrm{Eq}^X, \mathtt{refl}^X, {}^{-1_X}, *_X)$

$$x =_X x' :\Leftrightarrow \exists_{p \in Y}\big(p : x =_X x'\big),$$

$$\theta^{xx'}(p) :\Leftrightarrow p : x =_X x' \qquad \text{extensional}$$

$$\mathrm{Eq}^X : X \times X \rightsquigarrow \mathbb{V}_0$$

$$\mathrm{Eq}^X(x, x') := \{p \in Y \mid p : x =_X x'\}$$

$$\mathtt{refl}^X : \bigcurlywedge_{x \in X} \mathrm{Eq}^X(x, x)$$

$$^{-1_X} : \bigcurlywedge_{x, x' \in X} \mathbb{F}\Big(\mathrm{Eq}^X(x, x'), \mathrm{Eq}^X(x', x)\Big)$$

$$*_X : \bigcurlywedge_{x, x', x'' \in X} \mathbb{F}\Big(\mathrm{Eq}^X(x, x') \times \mathrm{Eq}^X(x', x''), \mathrm{Eq}^X(x, x'')\Big)$$

such that the following hold:

(ML$_1$) $\texttt{refl}_x * p =_{\mathrm{Eq}^X(x,x')} p$ and $p * \texttt{refl}_y =_{\mathrm{Eq}^X(x,x')} p$, for every $p \in \mathrm{Eq}^X(x,x')$.

(ML$_2$) $p * p^{-1} =_{\mathrm{Eq}^X(x,x)} \texttt{refl}_x$ and $p^{-1} * p =_{\mathrm{Eq}^X(y,y)} \texttt{refl}_y$, for every $p \in \mathrm{Eq}^X(x,x')$.

(ML$_3$) $(p * q) * r =_{\mathrm{Eq}^X(x,x''')} p * (q * r)$, for every $p \in \mathrm{Eq}^X(x,x'), q \in \mathrm{Eq}^X(x',x'')$ and $r \in \mathrm{Eq}^X(x'',x''')$.

(ML$_4$) If $p, q \in \mathrm{Eq}^X(x,x')$ and $r, s \in \mathrm{Eq}^X(x',x'')$ such that $p =_{\mathrm{Eq}^X(x,x')} q$ and $r =_{\mathrm{Eq}^X(x',x'')} s$, then $p * r =_{\mathrm{Eq}^X(x,x'')} q * s$.

A non-trivial example of a Martin-Löf set is $\texttt{Fam}(I, X)$, the set of families of subsets of the set X indexed by the set I.

Martin-Löf map $\widehat{f} := (f, f_1) \colon \widehat{X} \to \widehat{Y}$

$$f \colon X \to Y$$

$$f_1 : \bigcurlywedge_{x,x' \in X} \mathbb{F}\left(\mathrm{Eq}^X(x,x'), \mathrm{Eq}^Y(f(x), f(x'))\right).$$

such that

$$f_1(x, x, \texttt{refl}_x) =_{\mathrm{Eq}^Y(f(x),f(x))} \texttt{refl}_{f(x)}$$

If $x =_X x' =_X x''$, then

$$f_1(x, x'', p * q) =_{\mathrm{Eq}^Y(f(x),f(x''))} f_1(x, x', p) * f_1(x', x'', q)$$

Family of sets $\widehat{\Lambda} := (\lambda_0, \mathrm{Eq}^I, \lambda_2)$ over $\widehat{I}$

$$\lambda_2 : \bigcurlywedge_{(i,j)\in D(I)} \ \bigcurlywedge_{p\in \mathrm{Eq}^I(i,j)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \qquad (i,j) \in D(I),\ p \in \mathrm{Eq}^I(i,j),$$

(i) $\lambda_{ii}^{\mathtt{refl}_i} = \mathrm{id}_{\lambda_0(i)}$, $i =_I j =_I k$, $p \in \mathrm{Eq}^I(i,j), q \in \mathrm{Eq}^I(j,k)$ tfdc

$$\begin{array}{ccc} \lambda_0(i) & & \\ {\scriptstyle \lambda_{ij}^{p}}\downarrow & \searrow^{\lambda_{ik}^{p*q}} & \\ \lambda_0(j) & \xrightarrow[\lambda_{jk}^{q}]{} & \lambda_0(k). \end{array}$$

(ii) If $i =_I j$, then for for every $p \in \mathrm{Eq}^I(i,j)$ tfdc

$$\begin{array}{ccc} \lambda_0(i) & & \\ {\scriptstyle \lambda_{ij}^{p}}\downarrow & \searrow^{\mathrm{id}_{\lambda_0(i)}} & \\ \lambda_0(j) & \xrightarrow[\lambda_{ji}^{p^{-1}}]{} & \lambda_0(i) \end{array} \qquad \begin{array}{ccc} \lambda_0(j) & & \\ {\scriptstyle \lambda_{ji}^{p^{-1}}}\downarrow & \searrow^{\mathrm{id}_{\lambda_0(j)}} & \\ \lambda_0(i) & \xrightarrow[\lambda_{ij}^{p}]{} & \lambda_0(j). \end{array}$$

$$(i, x) =_{\hat{\sum}_{i\in I}\lambda_0(i)} (j, y) :\Leftrightarrow i =_I j \ \& \ \exists_{p\in \mathrm{Eq}^I(i,j)}\big(\lambda_{ij}^p(x) =_{\lambda_0(j)} y\big),$$

$$\Theta \in \widehat{\prod}_{i\in I}\lambda_0(i) :\Leftrightarrow \Theta \in \mathcal{DO}(I, \lambda_0) \ \& \ \forall_{p\in \mathrm{Eq}^I(i,j)}\big(\Theta_j =_{\lambda_0(j)} \lambda_{ij}^p(\Theta_i)\big).$$

If $\widehat{\Lambda} := (\lambda_0, \mathrm{Eq}^I, \lambda_2)$ is function-like over the Martin-Löf set $\widehat{I}$, then a structure of a Martin-Löf set is defined on $\hat{\sum}_{i\in I}\lambda_0(i)$.

The contractibility of the singleton type

Theorem

Let $\widehat{X}$ be a proof-relevant set, $x_0 \in X$ and let

$$\widehat{\texttt{PrfEql}}^{x_0} := (\mathrm{Eq}^{x_0}, \texttt{PrfEql}_1^{x_0})$$

the corresponding function-like family of sets over $\widehat{X}$. Let $\widehat{\sum}_{x \in X} \mathrm{Eq}^X(x, x_0)$ be equipped with its canonical structure of a Martin-Löf set. Then for every $(x, p) \in \widehat{\sum}_{x \in X} \mathrm{Eq}^X(x, x_0)$ we get

$$(x, p) =_{\widehat{\sum}_{x \in X} \mathrm{Eq}^X(x, x_0)} \big(x_0, \texttt{refl}_{x_0}\big).$$

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