8 valued non-deterministic semantics for modal logic

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Motivations

- Dugundji's result.

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- We do not like possible-world semantics.
- Non-deterministic semantics is more natural for weak-modal logics.

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5 $\Diamond\varphi \to \Box\Diamond\varphi$

- Axioms D and T are not that relevant for this talk.

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- Logic **H** is **K** minus the axiom K and minus D1 – D4.
- Logic **HD** is **H** plus D1 – D4.

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$\mathtt{M} = (\mathtt{Val}, \mathtt{D}, \mathtt{O})$ where:

1. $\mathtt{Val} = \{c_1, c_2, \ldots, c_n\}$: non-empty set of values.

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$v : \mathcal{L} \mapsto \mathtt{Val}$ such that, for each $\circ_j^i$,
$v(\circ_j^i(\varphi_1, \dots \varphi_i)) \in f_j^i(v(\varphi_1, \dots \varphi_i))$.

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Tautology

$\models_{\mathtt{M}} \varphi$ iff for any valuation v, $v(\varphi) \in \mathtt{D}$. The notion of consequence relation is as usual.

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- "Usual" deterministic semantics is a special case of non-deterministic semantics
- Logic **H** and **HD** do not have deterministic finitely valued semantics.
- NEC-free fragments of modal logics do not have natural semantics at all.

8-valued framework

We are interested in the following set of matrices:

- $\mathtt{Val} = \{\mathtt{T}_\Diamond, \mathtt{T}, \mathtt{t}_\Diamond, \mathtt{t}, \mathtt{f}, \mathtt{f}_\Diamond, \mathtt{F}, \mathtt{F}_\Diamond\}$.

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- $\texttt{D} = \{\texttt{T}_\Diamond, \texttt{T}, \texttt{t}_\Diamond, \texttt{t}\}$.

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- $\texttt{Val} = \{\texttt{T}_\Diamond, \texttt{T}, \texttt{t}_\Diamond, \texttt{t}, \texttt{f}, \texttt{f}_\Diamond, \texttt{F}, \texttt{F}_\Diamond\}$.
- $\texttt{D} = \{\texttt{T}_\Diamond, \texttt{T}, \texttt{t}_\Diamond, \texttt{t}\}$.
- $\texttt{O}$ interprets $\neg, \rightarrow, \Diamond, \Box$. The rest of the Boolean connectives are taken to be non-primitive.

A bit of philosophical story

Table: Meaning of values

Value	Status of the sentence
$\mathtt{T}_{\Diamond}$	$\Box\varphi, \Diamond\varphi, \varphi$ (necessary, possible and true)
$\mathtt{T}$	$\Box\varphi, \neg\Diamond\varphi, \varphi$ (necessary, not possible and true)
$\mathtt{t}_{\Diamond}$	$\neg\Box\varphi, \Diamond\varphi, \varphi$ (not necessary, possible and true)
$\mathtt{t}$	$\neg\Box\varphi, \neg\Diamond\varphi, \varphi$ (not necessary, not possible and true)
$\mathtt{f}_{\Diamond}$	$\neg\Box\varphi, \Diamond\varphi, \neg\varphi$ (not necessary, possible and false)
$\mathtt{F}$	$\Box\varphi, \neg\Diamond\varphi, \neg\varphi$ (necessary, not possible and false)
$\mathtt{F}_{\Diamond}$	$\Box\varphi, \Diamond\varphi, \neg\varphi$ (necessary, possible and false)
$\mathtt{f}$	$\neg\Box\varphi, \neg\Diamond\varphi, \neg\varphi$ (not necessary, not possible and false)

Two implications

$\to_{\mathbf{H}}$	$\mathtt{T}_{\Diamond}$	$\mathtt{T}$	$\mathtt{t}_{\Diamond}$	$\mathtt{t}$	$\mathtt{F}_{\Diamond}$	$\mathtt{F}$	$\mathtt{f}_{\Diamond}$	$\mathtt{f}$
$\mathtt{T}_{\Diamond}$	D	D	D	D	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$
$\mathtt{T}$	D	D	D	D	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$
$\mathtt{t}_{\Diamond}$	D	D	D	D	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$
$\mathtt{t}$	D	D	D	D	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$
$\mathtt{F}_{\Diamond}$	D	D	D	D	D	D	D	D
$\mathtt{F}$	D	D	D	D	D	D	D	D
$\mathtt{f}_{\Diamond}$	D	D	D	D	D	D	D	D
$\mathtt{f}$	D	D	D	D	D	D	D	D

Two implications

$\to_{\mathbf{K}}$	$\mathtt{T}_{\Diamond}$	$\mathtt{T}$	$\mathtt{t}_{\Diamond}$	$\mathtt{t}$	$\mathtt{F}_{\Diamond}$	$\mathtt{F}$	$\mathtt{f}_{\Diamond}$	$\mathtt{f}$
$\mathtt{T}_{\Diamond}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\{\mathtt{f}_{\Diamond},\mathtt{f}\}$	$\{\mathtt{f}_{\Diamond},\mathtt{f}\}$
$\mathtt{T}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\{\mathtt{f}_{\Diamond},\mathtt{f}\}$	$\{\mathtt{f}_{\Diamond},\mathtt{f}\}$
$\mathtt{t}_{\Diamond}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$
$\mathtt{t}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$
$\mathtt{F}_{\Diamond}$	$\mathtt{D}$	$\mathtt{D}$	$\{\mathtt{t}_{\Diamond},\mathtt{t}\}$	$\{\mathtt{t}_{\Diamond},\mathtt{t}\}$	$\mathtt{D}$	$\mathtt{D}$	$\{\mathtt{t}_{\Diamond},\mathtt{t}\}$	$\{\mathtt{t}_{\Diamond},\mathtt{t}\}$
$\mathtt{F}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$
$\mathtt{f}_{\Diamond}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$
$\mathtt{f}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$

Five negations

φ	$\neg$**D1**	$\neg$**D2**	$\neg$**D3**	$\neg$**D4**	$\neg$**D1234**
$\mathtt{T}_\Diamond$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\{\mathtt{f}_\Diamond, \mathtt{f}\}$	$\{\mathtt{F}, \mathtt{f}\}$	$\mathtt{f}$
$\mathtt{T}$	$\overline{\mathtt{D}}$	$\{\mathtt{F}_\Diamond, \mathtt{F}\}$	$\{\mathtt{F}_\Diamond, \mathtt{F}\}$	$\{\mathtt{F}, \mathtt{f}\}$	$\mathtt{F}$
$\mathtt{t}_\Diamond$	$\{\mathtt{F}_\Diamond, \mathtt{f}_\Diamond\}$	$\overline{\mathtt{D}}$	$\{\mathtt{f}_\Diamond, \mathtt{f}\}$	$\mathtt{D}$	$\mathtt{f}_\Diamond$
$\mathtt{t}$	$\{\mathtt{F}_\Diamond, \mathtt{f}_\Diamond\}$	$\{\mathtt{F}_\Diamond, \mathtt{F}\}$	$\mathtt{F}_\Diamond$	$\overline{\mathtt{D}}$	$\mathtt{F}_\Diamond$
$\mathtt{F}_\Diamond$	$\mathtt{D}$	$\mathtt{D}$	$\{\mathtt{t}_\Diamond, \mathtt{t}\}$	$\{\mathtt{T}, \mathtt{t}\}$	$\mathtt{t}$
$\mathtt{F}$	$\mathtt{D}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\{\mathtt{T}, \mathtt{t}\}$	$\mathtt{T}$
$\mathtt{f}_\Diamond$	$\{\mathtt{T}_\Diamond, \mathtt{t}_\Diamond\}$	$\mathtt{D}$	$\{\mathtt{t}_\Diamond, \mathtt{t}\}$	$\mathtt{D}$	$\mathtt{t}_\Diamond$
$\mathtt{f}$	$\{\mathtt{T}_\Diamond, \mathtt{t}_\Diamond\}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{T}_\Diamond$

Plenty of boxes and diamonds

φ	$\Diamond_\mathbf{4}$	$\Box_\mathbf{4}$	$\Diamond_\mathbf{5}$	$\Box_\mathbf{5}$	$\Diamond_\mathbf{B}$	$\Box_\mathbf{B}$
$\mathtt{T}_\Diamond$	$\mathtt{D}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\mathtt{D}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\mathtt{D}$
$\mathtt{T}$	$\overline{\mathtt{D}}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\overline{\mathtt{D}}$	$\mathtt{D}$	$\{\mathtt{F}_\Diamond, \mathtt{F}\}$	$\mathtt{D}$
$\mathtt{t}_\Diamond$	$\mathtt{D}$	$\overline{\mathtt{D}}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\overline{\mathtt{D}}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\overline{\mathtt{D}}$
$\mathtt{t}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\{\mathtt{F}_\Diamond, \mathtt{F}\}$	$\overline{\mathtt{D}}$
$\mathtt{F}_\Diamond$	$\mathtt{D}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\mathtt{D}$	$\mathtt{D}$	$\mathtt{D}$
$\mathtt{F}$	$\overline{\mathtt{D}}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\overline{\mathtt{D}}$	$\mathtt{D}$	$\overline{\mathtt{D}}$	$\mathtt{D}$
$\mathtt{f}_\Diamond$	$\mathtt{D}$	$\overline{\mathtt{D}}$	$\{\mathtt{T}_\Diamond, \mathtt{T}\}$	$\overline{\mathtt{D}}$	$\mathtt{D}$	$\overline{\mathtt{D}}$
$\mathtt{f}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$	$\overline{\mathtt{D}}$

Completeness and some observations

- The above semantics is strongly sound and complete with respect to the mentioned axiomatizations.

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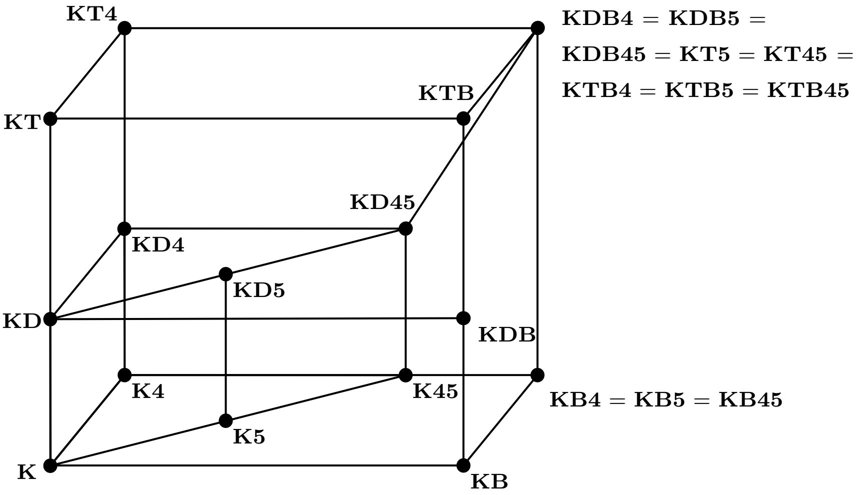
- NEC is missing.

Starting-point: well-known modal logics and their NEC-free fragments



Valuation Refinements

mth-level valuations

Let v be a valuation in an nmatrix $\mathtt{M}$ and $\mathtt{L}$ logic induced by $\mathtt{M}$. Let $\mathbf{Sp} \subseteq \mathtt{D}$ be the set of super-designated values. We say that v is:

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1. a 0th-level $\mathtt{L}$-valuation w.r.t $\mathbf{Sp}$ if v is an $\mathtt{L}$-valuation.
2. v is a $m+1$th-level $\mathtt{L}$-valuation w.r.t $\mathbf{Sp}$ if v is mth-level $\mathtt{L}$-valuation w.r.t $\mathbf{Sp}$ and v assigns a super-designated value to every formula φ that has a designated value for any mth-level $\mathtt{L}$-valuation v'.

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1. a 0th-level $\mathtt{L}$-valuation w.r.t $\mathbf{Sp}$ if v is an $\mathtt{L}$-valuation.
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3. We say that v is a $\mathtt{L}^{+}$-valuation w.r.t $\mathbf{Sp}$ iff v is a mth-level $\mathtt{L}$-valuation w.r.t $\mathbf{Sp}$ for every m.

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- If $\mathbf{Sp} = \{\mathtt{T}, \mathtt{T}_\Diamond\}$, then the resulting system is closed under NEC.

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Getting the NEC back

- If $\mathbf{Sp} = \{\mathtt{T}, \mathtt{T}_{\Diamond}\}$, then the resulting system is closed under NEC.
- If the initial logic validates $\mathsf{4}$, already the first level of valuations.
- At each level you regain NEC for the previous level.

Example

Consider a formula $\Box(\Box(p \to p))$ and the following subset of valuations:

#	p	$p \to p$	$\Box(p \to p)$	$\Box(\Box(p \to p))$
1	t	$\mathtt{T}_\Diamond$	$\mathtt{T}_\Diamond$	$\mathtt{T}_\Diamond$
2	t	$\mathtt{T}_\Diamond$	$\mathtt{T}_\Diamond$	T
3	t	$\mathtt{T}_\Diamond$	T	$\mathtt{T}_\Diamond$
4	t	$\mathtt{T}_\Diamond$	T	T
5	t	T	$\mathtt{T}_\Diamond$	$\mathtt{T}_\Diamond$
6	t	T	$\mathtt{T}_\Diamond$	T
7	t	T	T	$\mathtt{T}_\Diamond$
8	t	T	T	T
9	t	t	$\mathtt{F}_\Diamond$	$\mathtt{T}_\Diamond$
10	t	t	$\mathtt{F}_\Diamond$	T
11	t	t	F	$\mathtt{T}_\Diamond$
12	t	t	F	T
13	t	t	f	$\mathtt{f}_\Diamond$
14	t	t	f	f
15	t	t	f	$\mathtt{F}_\Diamond$
16	t	t	f	F
17	t	t	$\mathtt{f}_\Diamond$	$\mathtt{f}_\Diamond$
18	t	t	$\mathtt{f}_\Diamond$	f
19	t	t	$\mathtt{f}_\Diamond$	$\mathtt{F}_\Diamond$
20	t	t	$\mathtt{f}_\Diamond$	F

Example continue

If we look only at the first level valuations we are left with:

#	p	$p \to p$	$\Box(p \to p)$	$\Box(\Box(p \to p))$
1	t	$T_\Diamond$	$T_\Diamond$	$T_\Diamond$
2	t	$T_\Diamond$	$T_\Diamond$	T
3	t	$T_\Diamond$	T	$T_\Diamond$
4	t	$T_\Diamond$	T	T
5	t	T	$T_\Diamond$	$T_\Diamond$
6	t	T	$T_\Diamond$	T
7	t	T	T	$T_\Diamond$
8	t	T	T	T

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- The lack of decidability. Solved for some specific cases.

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- What other logics can be defined in this framework? [Work in progress]
- On what does the valuation hierarchy depend?
- What about other principles weaker than NEC?
- What modal logics cannot be described by these semantics?

Thank you for your attention!