

BITCOIN BACKBONE, CONSENSUS, VARIABLE DIFFICULTY

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Bitcoin info

- Bitcoin was the **first decentralized cryptocurrency** with no need for a trusted central authority.
 - **Previous work:** Pricing functions of Dwork and Naor [1992], MicroMint of Rivest and Shamir [1996], Hashcash of Back [1997,2002], Szabo's bit gold [1998], Karma by Vishnumurthy, Chandrakumar, Sirer [2003].
- Introduced in the 2008 paper "Bitcoin: A Peer-to-Peer Electronic Cash System" by **Satoshi Nakamoto** (a pseudonym).
- Released as **open-source code** in 2009; first block: **9, Jan 2009**.
 - Nowadays there are more than than **700,000** blocks.
- The total number of bitcoins will not exceed **21 million** and this limit is expected to be reached around **2140**.
 - Nowadays there are more than **19 million** bitcoins in circulation.
 - The smallest denomination is the **satoshi**, equal to **10^{-8} bitcoins**.

Bitcoin: a solution to two problems

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- Bitcoin was a fresh solution at an **old, fundamental, and well-studied** problem in distributed computing, the **consensus problem**.

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Formal analysis

- A **formal** description of the **model** in which the problem and its solution can be described.
- The **properties** that a suggested solution should satisfy.
- A **formal** description of the protocol.
- **Proof** that Bitcoin backbone indeed has the desired properties.

The model

- **Synchronous** model: time is discrete and divided in **rounds**.
- A number of honest parties n and an adversary that controls t parties.
 - Honest parties act **independently**.
 - Parties controlled by the adversary **collaborate**.
- Parties communicate by **broadcasting** a message.

The **adversary** can:

- **inject** messages into a party's incoming messages.
- **reorder** a party's incoming messages.
- **Anonymous** setting: parties cannot associate a message to a sender; they don't even know if two messages come from the same sender.

What is not in the model

- Honest parties **losing** messages or becoming **eclipsed** or becoming unable to know the current **time**.
 - Parties experiencing such issues are factored into the **adversary**.
- The honest parties' **incentives**.
 - On the other hand, **adversarial** parties wish to inflict the worst possible damage independently of utility.
- An **adversary** with computational power that even on occasion, exceeds that of honest parties.
- Attacks that exploit specific weaknesses of the underlying cryptographic primitives.

[We will use idealized versions of hash functions and digital signatures].

Hash functions

A **cryptographic hash function** is a **deterministic** algorithm

$$H : \{0, 1\}^* \rightarrow \{0, 1\}^K$$

with the following properties.

- **Preimage resistance:** Given $y \in \{0, 1\}^K$ it should be computationally infeasible to compute x such that $H(x) = y$.
- **Second-preimage resistance:** Given x and $y = H(x)$ it should be computationally infeasible to compute a $x' \neq x$ such that $H(x') = y$.
- **Collision resistance:** It should be computationally infeasible to compute $x \neq x'$ such that $H(x) = H(x')$.

For a meaningful formal definition one considers cryptographic hash **families**.

Proof-of-work in the random-oracle model

A moderately hard computational task: Given a hash-function $H(\cdot)$ with range $\{0, 1\}^k$ and a y , find x such that $H(x, y)$ begins with a lot of zeroes. More generally, given a target T ,

- find x such that $H(x, y) < T$.

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We'll work in the "random oracle" model. That is, we assume the existence of a hash-function $H(\cdot)$ that operates as follows.

- On a query x , the returned value $H(x)$ is a random number from the range of $H(\cdot)$, unless x has been queried before in which case $H(\cdot)$ is consistent (equal to the previous returned value).

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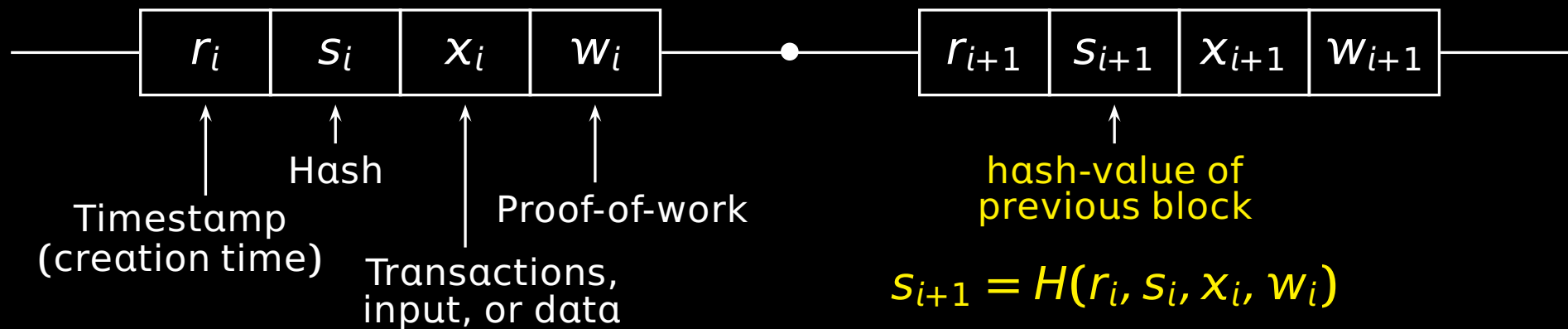
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- On a query x , the returned value $H(x)$ is a random number from the range of $H(\cdot)$, unless x has been queried before in which case $H(\cdot)$ is consistent (equal to the previous returned value).
- A query is successful with probability $\frac{T}{2^k}$, and one needs in expectation $\frac{2^k}{T}$ calls to the oracle $H(\cdot)$ for a proof-of-work.
- Among $\text{poly}(k)$ queries, the probability of a collision (two distinct x and x' with $H(x) = H(x')$) is exponentially small in k .

Bitcoin's data structure: the blockchain



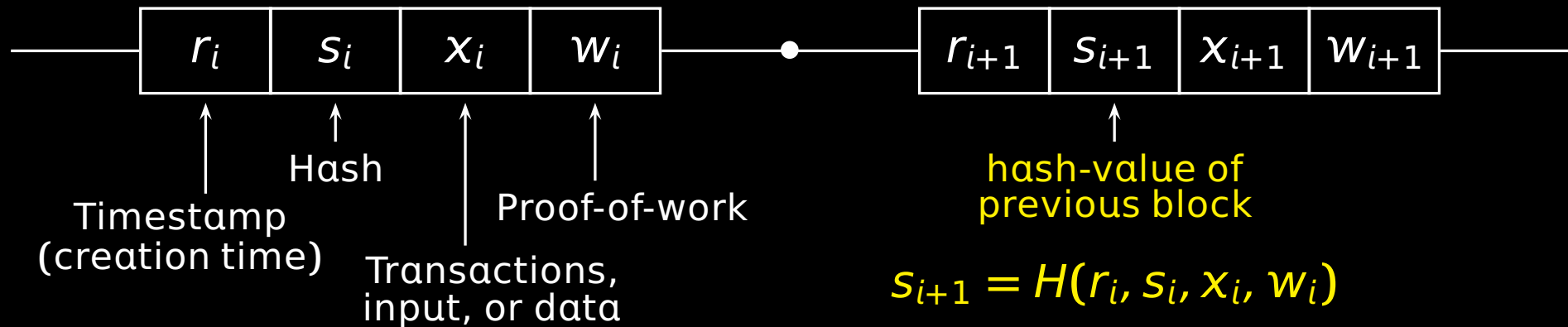
- A **block** (r, s, x, w) is **valid** if it has a **small hash-value**, providing a **proof-of-work**:

$$H(r, s, x, w) < T.$$

- A **chain** is **valid** if all its blocks provide a proof-of-work and each block **extends** the previous one:

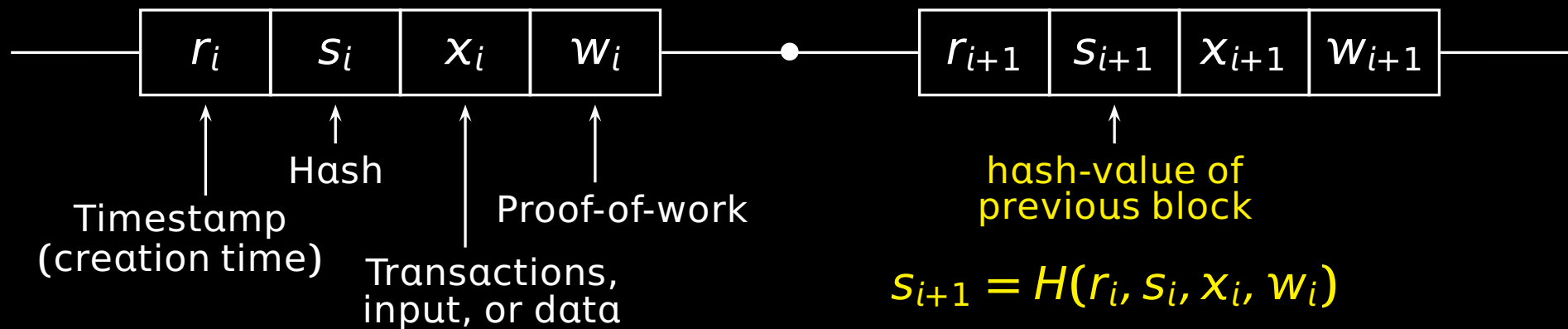
$$\text{for each } i, \quad s_{i+1} = H(r_i, s_i, x_i, w_i) \text{ and } r_{i+1} > r_i.$$

Comments on the blockchain



- To alter the contents of a block and preserve the length of the chain the adversary either has to discover a collision in $H(\cdot)$ or compute all the subsequent blocks.
 - Thus the adversary *cannot* delete, copy, inject, or predict blocks.
- By adjusting the target T we control how hard is computing a block: the lower the target the higher the difficulty, $w \log 1/T$.

Transactions on the blockchain



A transaction has the following form:

- “From the output (say **10BTC**) of transaction i in block j (which was sent to public pk_0), send **2BTC** to pk_1 and **7BTC** to pk_2 ” --- signed with sk_0 .
- Fees, coinbase transaction.
- Parties need to **agree** on which is the j -th block.

Bitcoin backbone: A distributed randomized algorithm

In each round r , each party with a chain C_0 performs the following:

- **Receive** from the network (block)chains C_1, C_2, \dots
- Choose the **first longest** chain C among the **valid** ones in $\{C_0, C_1, C_2, \dots\}$. (Order matters*.)
- Try to extend the **longest** chain C .

This is modeled by a **Bernoulli trial** with a probability of success that depends on the target T .

- Suppose its last block is the i -th one and equal to (r_i, s_i, x_i, w_i) with $s = H(r_i, s_i, x_i, w_i)$. Find $w \in \{1, 2, \dots, q\}$ such that

$$H(r, s, x, w) < T.$$

If successful, let $C \leftarrow C \parallel (r, s, x, w)$.

- If $C \neq C_0$ (i.e., you computed or switched-to another (longer) chain), **diffuse** the new chain C .

An execution example

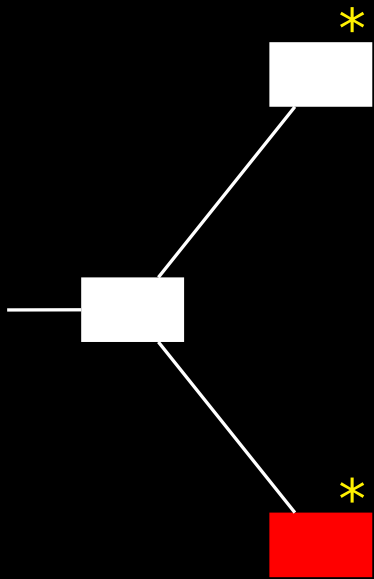
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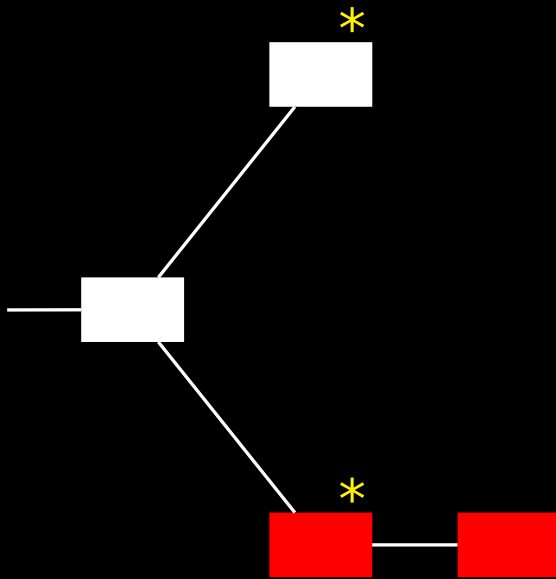
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- A **star (*)** on a block means that an honest party **has** the chain ending with that block at the given round.

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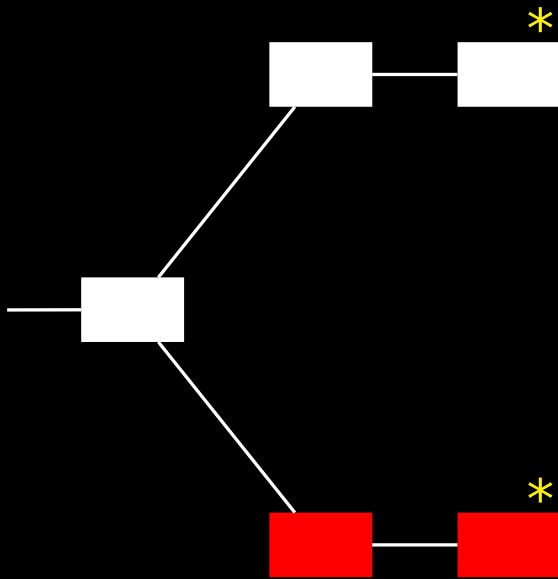
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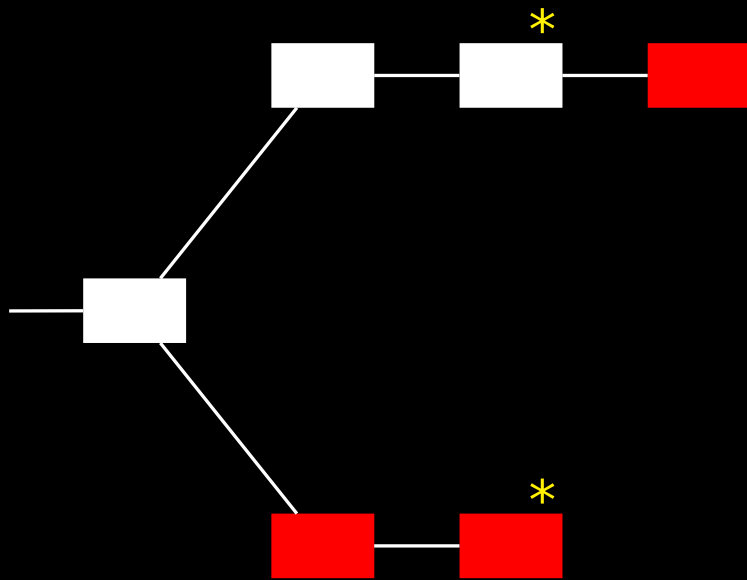
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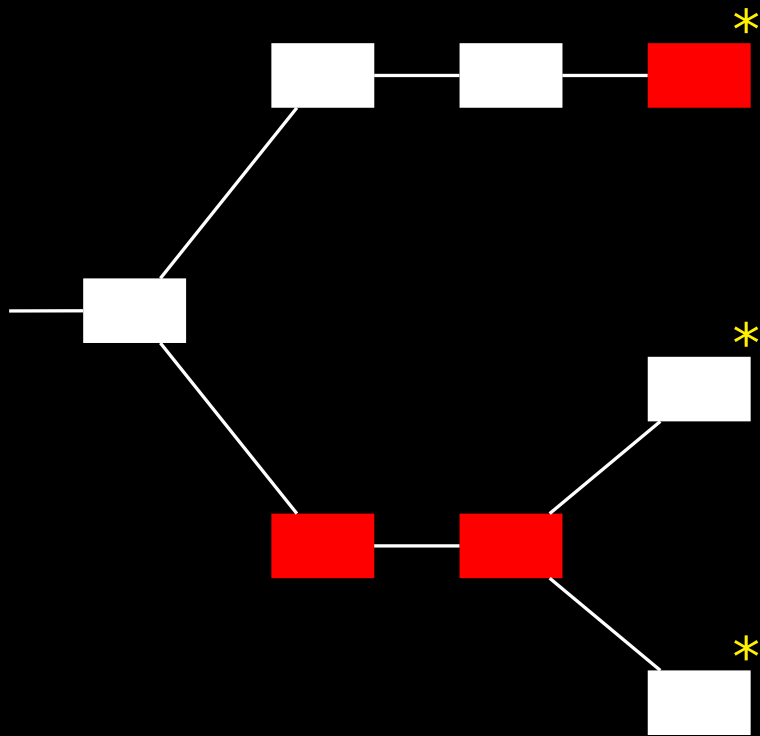
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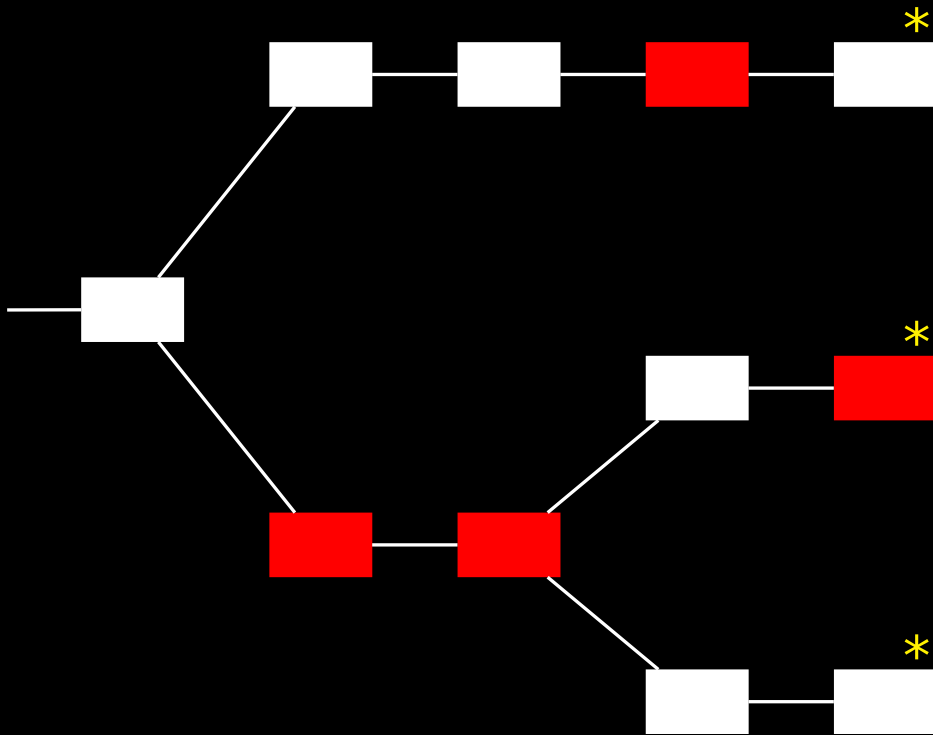
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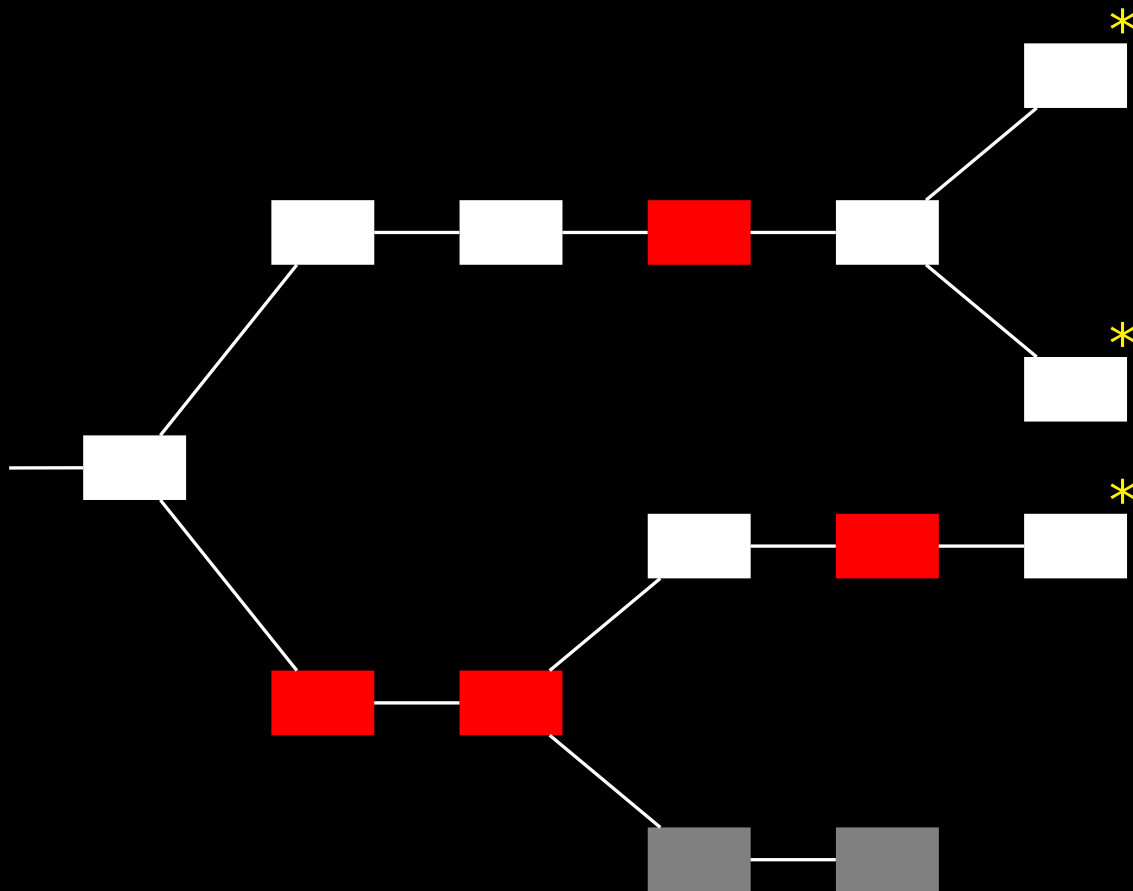
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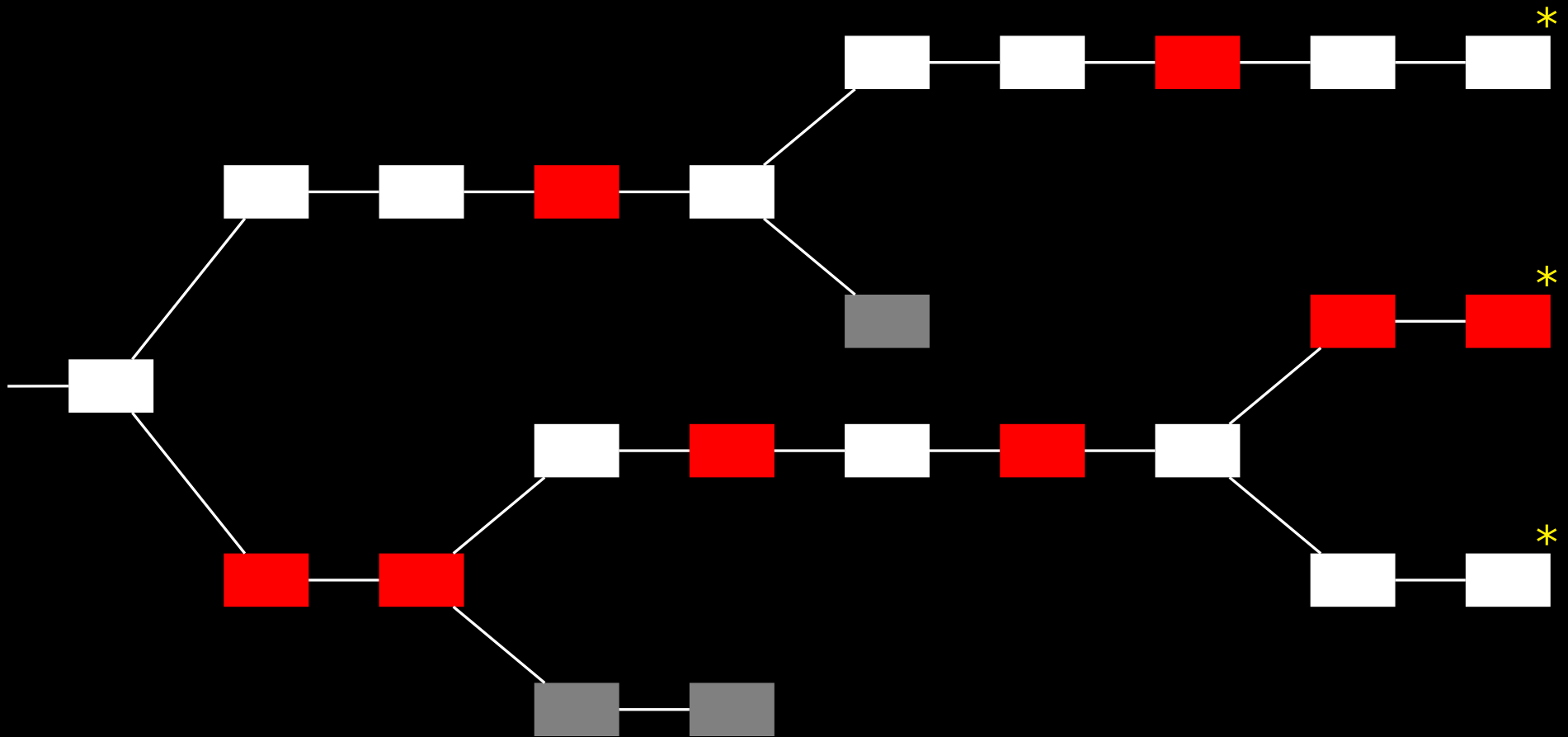
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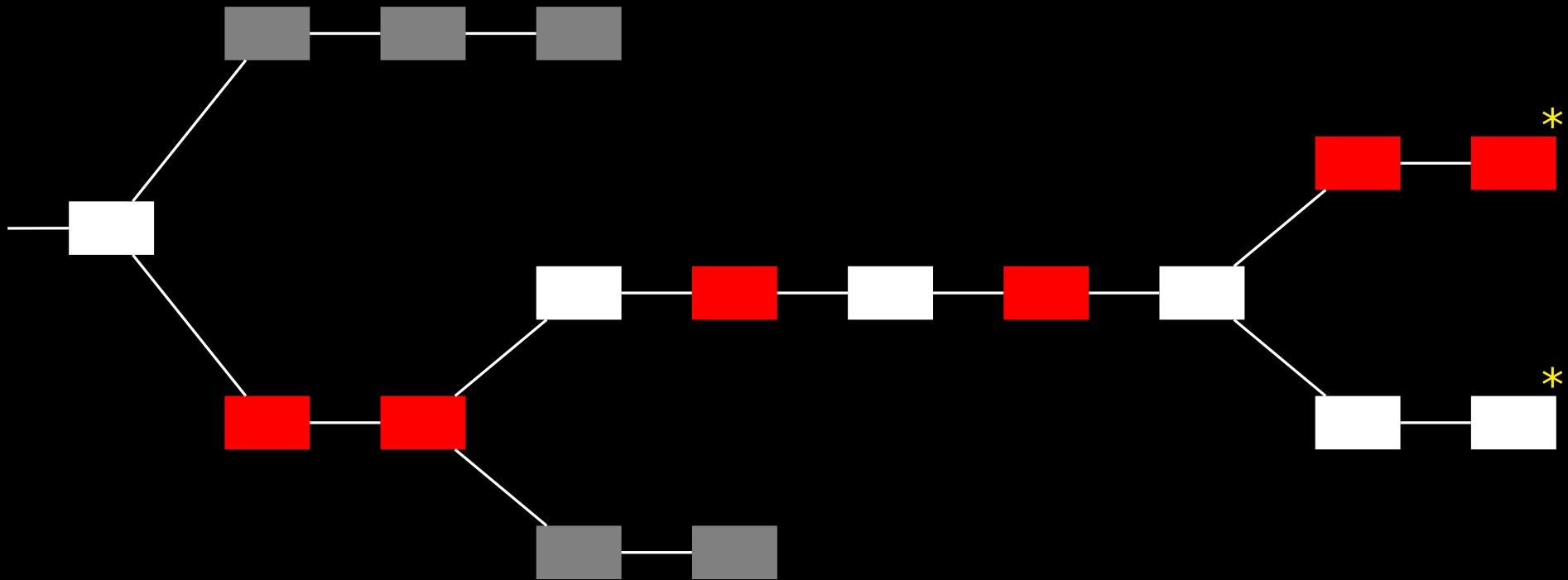
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Persistence. If a transaction is confirmed by an honest party, no honest party will ever disagree about the position of that transaction in the ledger.

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Chain-Growth Property. The chain of any honest party grows at least at a steady rate.

Analysis: Random Variables

Successful Round. A round r in which **at least one** honest party computes a block.

$X_r = 1 \iff r$ is a **successful** round

$$f := \mathbf{E}[X_r] = 1 - (1 - p)^n \approx pn$$

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Adversary. For each **query** j ,

$Z_j = 1 \iff$ the adversary computed a block with his j -th query

$$\mathbf{E}[Z_r] = \mathbf{E}[Z_1 + \dots + Z_t] = \mathbf{E}[Z_r] = \mathbf{E}[Z_1] + \dots + \mathbf{E}[Z_t] = pt$$

Chain-Growth Lemma

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Chernoff Bound. *Suppose $\{X_i : i \in [n]\}$ are mutually independent Boolean random variables, with $\Pr[X_i = 1] = p$, for all $i \in [n]$. Let $X = \sum_{i=1}^n X_i$ and $\mu = pn$. Then, for any $\delta \in (0, 1]$,*

$$\Pr[X \leq (1 - \delta)\mu] \leq e^{-\delta^2\mu/2} \quad \text{and} \quad \Pr[X \geq (1 + \delta)\mu] \leq e^{-\delta^2\mu/3}.$$

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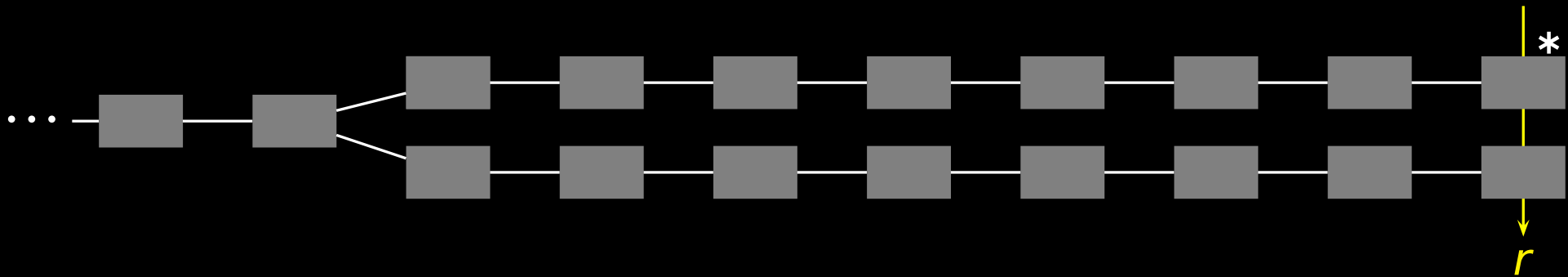
Chain-growth property. With probability at least $1 - e^{-\Omega(\epsilon^2 fs)}$, the chain of any honest party increases by at least

$$(1 - \epsilon)fs \approx (1 - \epsilon)pns$$

blocks after s consecutive rounds. ($\mathbf{E}[X_1 + \dots + X_s] = fs \approx pns$.)

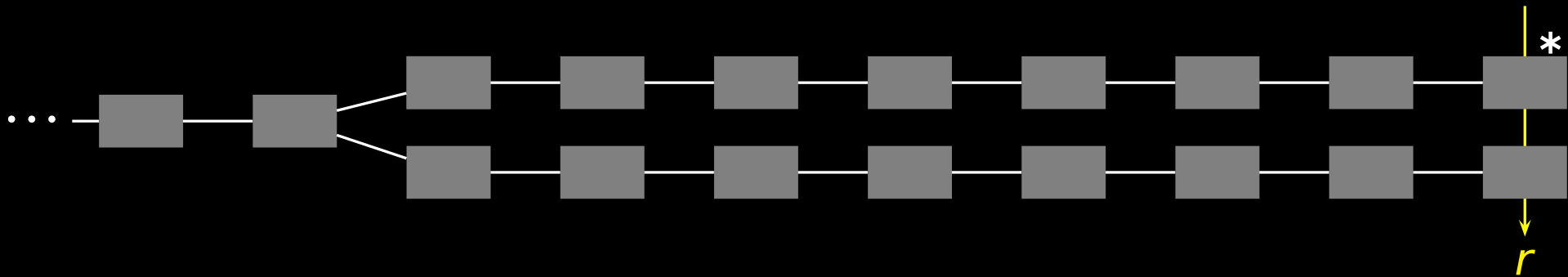
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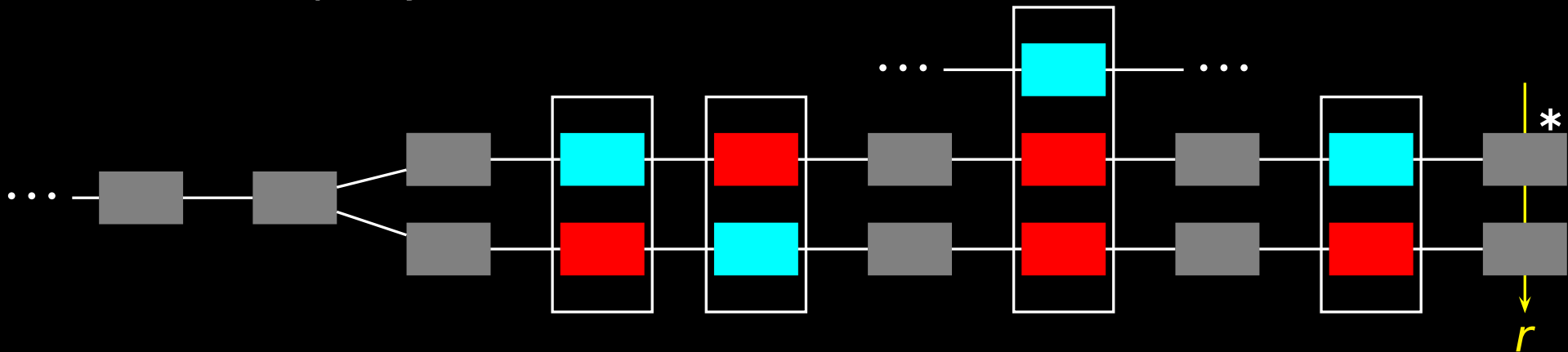
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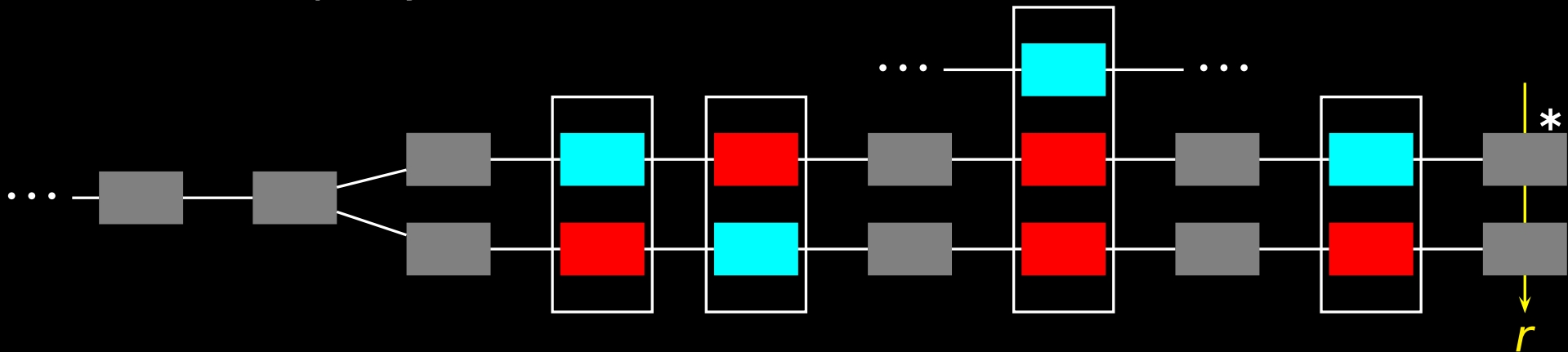
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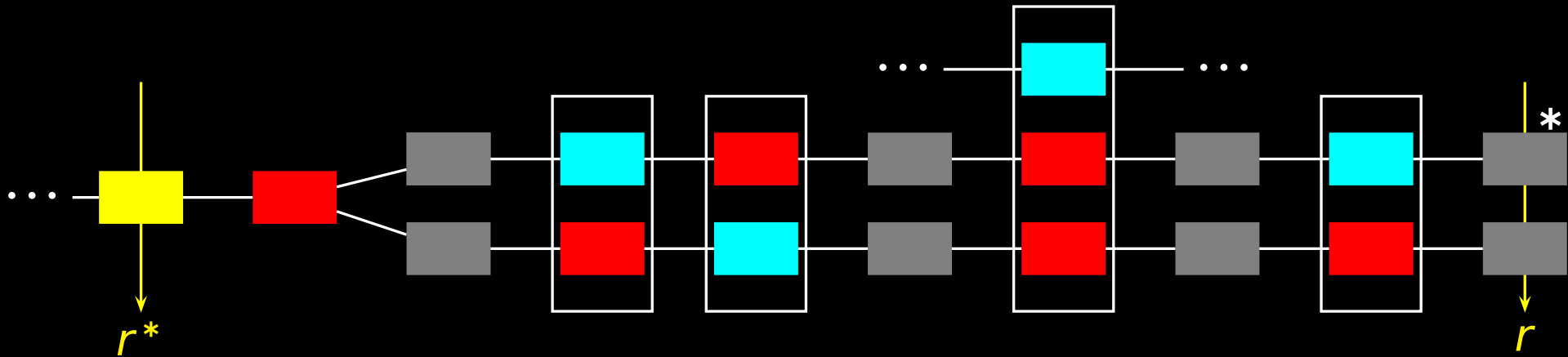


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Proof. Suppose a block of height ℓ was computed by an honest party at a round u with $Y_u = 1$. If any honest party computed a block of height ℓ at any round $r < u$, then any honest party is trying to extend a chain of length at least ℓ at round u . Similarly for $r > u$.

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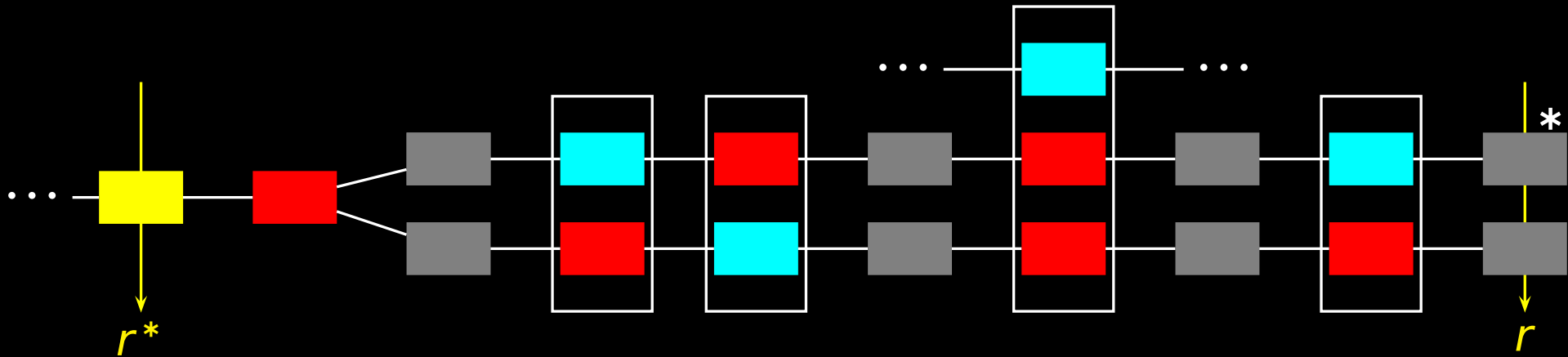
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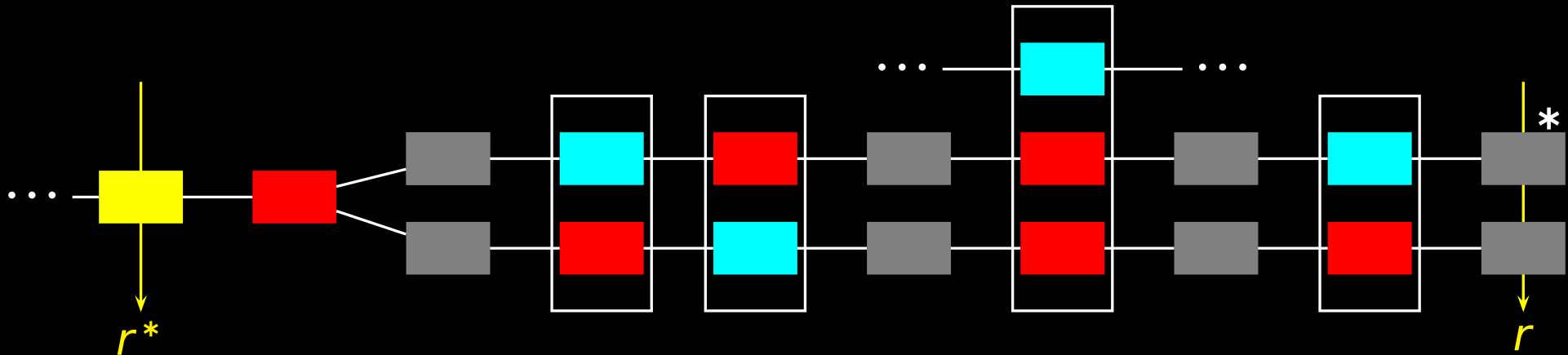
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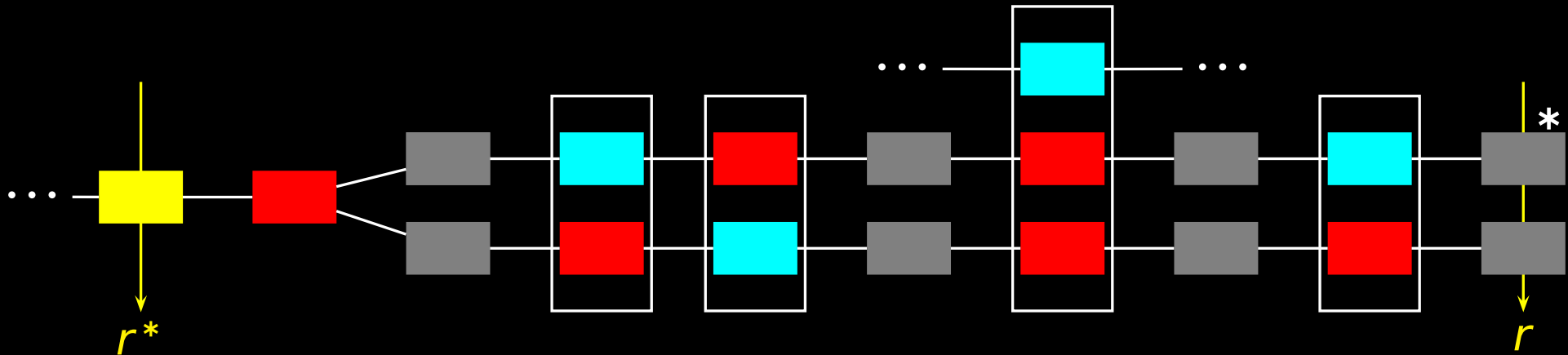


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$$E[\sum Y_i] \approx pn(1-f)|S|$$

$$E[\sum Z_i] = pt|S|.$$

Proof of the common-prefix lemma (cont'd)

Recall that $\mathbf{E}[Y_i] > f(1 - f)$. Let $Y(S) = \sum_{r \in S} Y_r$. Then, since $\mathbf{E}[Y(S)] = \sum_{r \in S} f(1 - f) = f(1 - f)|S|$, by the Chernoff bound,

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Assuming these bad events don't occur (union bound) and the Honest Majority Assumption

$$\begin{aligned} Z(S) &< (1 + \epsilon)pt|S| \\ &< (1 + \epsilon)(1 - \delta)pn|S| && (t < (1 - \delta)n) \\ &< (1 + \epsilon)(1 - \delta) \cdot \frac{f}{1 - f} \cdot |S| && (1 - f)pn < f \\ &< (1 - \epsilon)f|S| && (\delta > 3\epsilon + 3f) \\ &< Y(S) \end{aligned}$$

□

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Compare to
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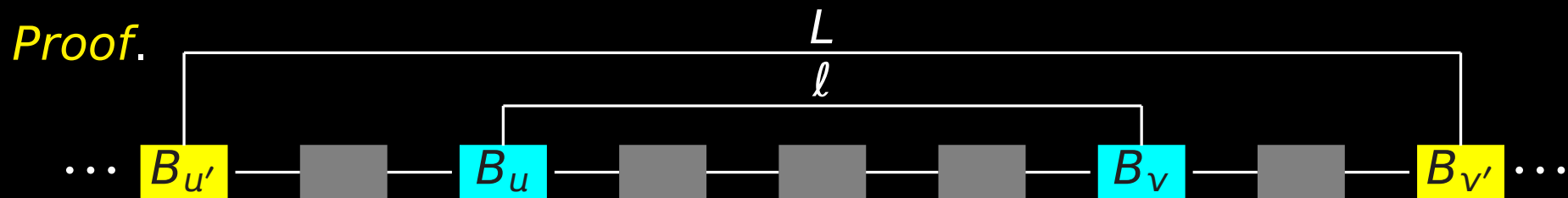
Corollary. If $t < (1 - \epsilon)n$, there is at least one honest block among any ℓ consecutive blocks in the chain of an honest party.

Proof. The ratio of adversarial blocks is less than $(1 + \epsilon)(1 - \epsilon) < 1$.

Chain Quality

Chain Quality. For any ℓ (sufficiently many) blocks in the chain of an honest party, the ratio of adversarial blocks is at most

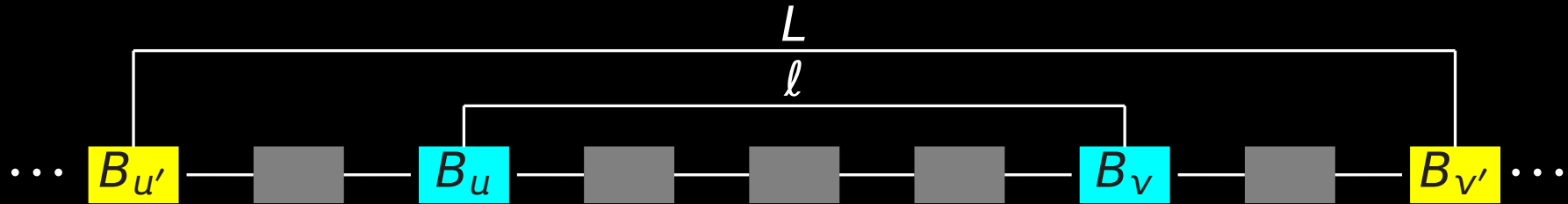
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- u' is greatest such that $B_{u'}$ was computed by an honest party.
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- $S = \{r : r_1 \leq r < r_2\}$.

Proof of Chain-Quality Property

Proof Cont'd.



We may assume that all the L blocks have been computed during the rounds in the set S .

- The number of successful rounds is at least $X \geq (1 - \frac{\epsilon}{3})pn|S|$.
- The number of adversarial blocks is at most $Z \leq (1 + \frac{\epsilon}{3})pt|S|$.
- Chain growth implies that $L \geq X$.
- The fraction of adversarial blocks is at most

$$\frac{Z}{L} \leq \frac{Z}{X} \leq \frac{1 + \frac{\epsilon}{3}}{1 - \frac{\epsilon}{3}} \cdot \frac{t}{n} \leq (1 + \epsilon) \cdot \frac{t}{n}.$$

□

Tightness of Chain Quality

Theorem. *There exists an adversary such that, with probability at least $1 - e^{-\Omega(\epsilon^2 \ell)}$ ($\ell = \Omega(1/\epsilon)$), there will be ℓ consecutive blocks in the chain of every honest party in which the fraction of adversarial blocks is at least*

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A selfish mining attack.

- The adversary keeps on extending a private chain.
- Whenever an honest party finds a solution, the (rushing) adversary releases one block from the private chain.
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- If the private chain is depleted the adversary returns to the public chain.

Assumption. Ties between chains of equal length always favor the adversary.

Analysis of the Selfish Mining Attack

- Consider a set S of at least $\ell/(1-\epsilon)pn$ consecutive rounds.
 - This implies $X(S) \leq \ell$ (recall Chain-Growth Property).
- The number Z of adversarial blocks is at least $\frac{t}{n} \cdot \ell$.
- The number Z' of **orphaned adversarial blocks** computed in S is at most $\epsilon\ell$ with high probability.
 - k adversarial blocks may be orphaned, only if an honest party computes $k + 1$ sequential blocks.
- The number Z'' of adversarial blocks not released in S is at most $\epsilon^2\ell$ with high probability.
 - k adversarial blocks are not released, only if no honest party computed a block in the meantime.

The ratio of adversarial blocks is at least

$$\frac{Z - Z' - Z''}{X} \geq \frac{\frac{t}{n} \cdot \ell - \epsilon\ell - \epsilon^2\ell}{\ell} \geq \frac{t}{n} - 2\epsilon$$

□

Byzantine agreement (consensus)

A set of parties $\{1, \dots, n\}$, t of which are controlled and coordinated by an **adversary**. Parties have inputs $x_1, \dots, x_n \in \{0, 1\}$ and want to decide on outputs v_1, \dots, v_n so that the following conditions are satisfied.

- **Agreement:** All honest parties decide on the same value (i.e., if i and j are honest, then $v_i = v_j$).
- **Validity:** If all honest parties have the same input value x , then all honest parties **decide** x (i.e., if i is honest, then $v_i = x$).
- **Termination:** All honest processes should terminate.

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Remark. Note that n here is the **total** number of parties.

Byzantine Agreement: Fundamental Results

- One of the classical problems in distributed computing, a variant of which was first introduced in “Reaching Agreement in the Presence of Faults” [Pease-Shostak-Lamport 1980].
- Requires $n > 3t$, unless cryptography is used [PSL].
- Even with cryptographic tools, at least $t + 1$ rounds are needed [Fischer-Lynch and Dolev-Strong 1982].
- In an **asynchronous** or **anonymous** network no deterministic protocol exists [Fischer-Lynch-Paterson 1985]. But possible with **probability 1** [Ben-Or 1983].
- Bit complexity is $\Omega(nt)$ [Dolev-Reischuk 1985].
- **Fully Polynomial:** There exists a protocol for all $t < \frac{n}{3}$, that terminates in $t+1$ rounds, and both computation and communication are polynomial in n . [Garay, Moses, “Fully polynomial Byzantine agreement for $n > 3t$ processors in $t + 1$ rounds.” 1998]

Byzantine Agreement: Toy Proof

When **1** party out of n might be Byzantine, at least **2** rounds are needed.

- Upon receiving **00...001**, an honest party should output **0**.
 - Because of validity, since party p_n could be Byzantine.
- Upon receiving **00...011**, an honest party should output **0**.
 - Because party p_{n-1} could be Byzantine, and some parties might have received **00...001** and going to answer **0**.
- Upon receiving **00...0111**, an honest party should output **0**.
 - Because party p_{n-2} could be Byzantine, and some parties might have received **00...011** and going to answer **0**.
- ⋮
- Upon receiving **01...111**, an honest party should output **0**.

Contradiction! Because the first party could be Byzantine.

Consensus: $t < n/2$ necessary (even with crypto)

Proof. On input $0 \dots 0, 1 \dots, 1$, where there are $n/2$ zeroes and $n/2$ ones and all parties are honest, the protocol terminates in one of the following three states.

- A. All honest parties output 0.
- B. All honest parties output 1.
- C. Honest parties have mixed outputs.

The adversary chooses a strategy as follows.

- In case A, he corrupts the first half of parties and behaves honestly. **Validity** fails.
- In case B, he corrupts the second half of parties and behaves honestly. **Validity** fails.
- In case C, he does not corrupt any party. **Agreement** fails.

Consensus: $t < n/3$ necessary with delays

<p>$A(\text{mute})$</p> <p>$B(0) \quad C(0)$</p>	<p>Since B and C have 0 and A might have crashed, at some time t_A parties B and C should terminate with 0.</p>
<p>$A(1)$</p> <p>$B(\text{mute}) \quad C(1)$</p>	<p>Since A and C have 1 and B might have crashed, at some time t_B parties A and C should terminate with 1.</p>
<p>$A(1)$</p> <p>$B(0) \quad C(*)$</p>	<p>Adversary C talks to A as if he has a 1 and to B as if he has a 0. Meanwhile, he holds messages $A \leftrightarrow B$ for $t_C > t_A + t_B$ rounds.</p>

Nakamoto's insight

Re: Bitcoin P2P e-cash paper

Satoshi Nakamoto | Thu, 13 Nov 2008 19:34:25 -0800

James A. Donald wrote:

- > It is not sufficient that everyone knows X. We also
- > need everyone to know that everyone knows X, and that
- > everyone knows that everyone knows that everyone knows X
- > - which, as in the Byzantine Generals problem, is the
- > classic hard problem of distributed data processing.

The proof-of-work chain is a solution to the Byzantine Generals' Problem. I'll try to rephrase it in that context.

A number of Byzantine Generals each have a computer and want to attack the King's wi-fi by brute forcing the password, which they've learned is a certain number of characters in length. Once they stimulate the network to generate a packet, they must crack the password within a limited time to break in and erase the logs, otherwise they will be discovered and get in trouble. They only have enough CPU power to crack it fast enough if a majority of them attack at the same time.

They don't particularly care when the attack will be, just that they all agree.

It has been decided that anyone who feels like it will announce a time, and whatever time is heard first will be the official attack time. The problem is

<https://www.mail-archive.com/cryptography@metzdowd.com/msg09997.html>

Byzantine Agreement Protocol

Theorem [GKL2015]. Assuming $t < n/3$, the following protocol terminates after $\Theta(k)$ rounds in expectation and solves consensus with probability at least $1 - e^{-\Omega(k)}$.

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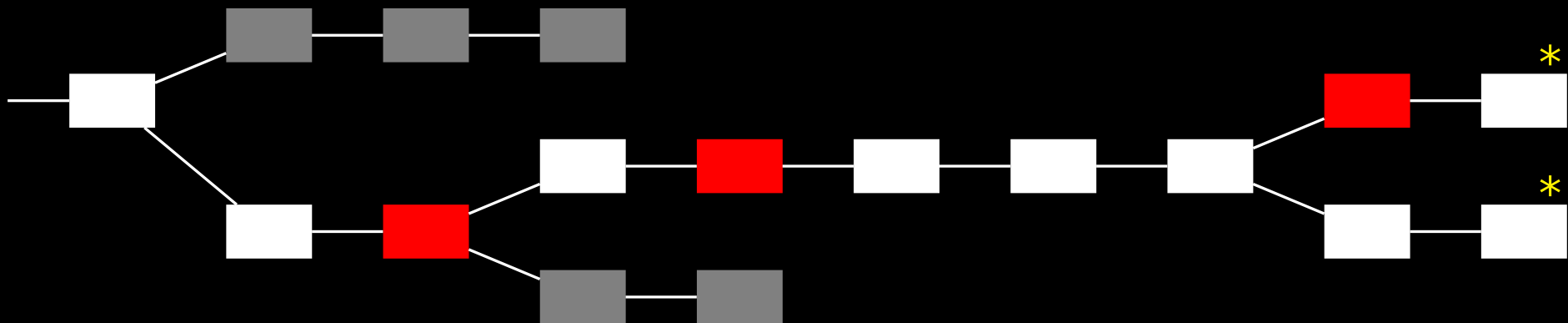
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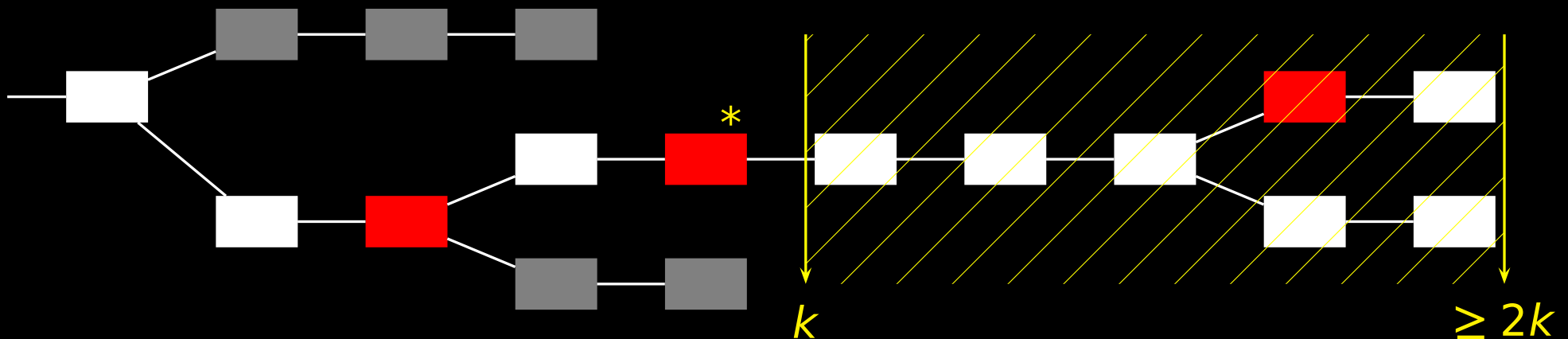
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Proof of Agreement and Validity

- By the common-prefix property, if the adversary has **less than half** of the total computational power, **Agreement** is satisfied with high probability.

This is because every honest party will output the majority of the input-bits included in the common prefix of their (possibly different) chains. (Consider the first time an honest party has a chain of length at least $2k$.)

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- By the chain-quality property, if the adversary has **less than one third** of the total computational power, **Validity** is satisfied with high probability.

This is because out of the k bits of the common prefix, the adversary has computed less than half of them. Therefore, if all the honest parties have the same input x , the majority of the bits in the common prefix will be x .

2-for-1 PoWs

Idea. Two kinds of blocks with a single query. Recall $H : \{0, 1\}^* \rightarrow \{0, 1\}^k$.

- **Normal blocks:** $H(x) < T = 2^a$.
- **Input blocks:** $[H(x)]^R < T' = 2^b$.

Here, $[y]^R$ is the number with binary expansion the reverse of y .

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Observation. As long as $a + b > \kappa$, the probabilities of obtaining a block of each kind are independent.

Proof. Let U random over $\{0, 1\}^\kappa$. Conditioning on $U < T$ leaves the a least significant bits of U random, while fixing the remaining $\kappa - a$ bits. Thus, the $a > \kappa - b$ most significant bits of U^R are random. It follows that

$$\Pr[U^R < T' | U < T] = \frac{2^{a-(\kappa-b)}}{2^a} = \frac{2^b}{2^\kappa} = \frac{T'}{2^\kappa} = \Pr[U^R < T'].$$

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- 1) Parties run the Bitcoin protocol, putting their own input-bit in every **input** block they compute and referencing all input blocks from the **normal** blocks they compute.
- 2) When they obtain a chain with at least $\frac{3k}{\delta} + 2k$ **normal** blocks they halt (after they diffuse it).
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Agreement follows from Common-Prefix Property because at least k blocks are pruned.

Proof for validity (sketch)

- Let C denote the prefix of the first $\frac{3k}{\delta} + 2k$ normal blocks.
- By Chain-Quality Property, the last k of C contain an honest normal block B , computed at some round r .
- Note that B contains all honest input blocks computed in $S = \{1, 2, \dots, r\}$. Let $X(S)$ denote their number and $Z(S)$ the adversarial input blocks.
- Thus,

$$\frac{Z}{X} < \frac{(1 + \epsilon)pt|S|}{(1 - \epsilon)f|S|} < \frac{(1 + \epsilon)(1 - \delta)pn|S|}{(1 - \epsilon)(1 - f)pn|S|} \leq \frac{(1 + \epsilon)(1 - \delta)}{(1 - \epsilon)(1 - f)} < 1$$

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$X'_i = 1$ if $X_i = 1$ and $X_j = 0$ for $i - \Delta < j < i$

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Remark. These definitions are not tight. In particular, we could do with a set of uniquely successful rounds such that any two are Δ -far away from each other.

Chain-Growth Lemma

Chain-Growth Lemma. *Suppose that at round u an honest party has a chain of length ℓ . Then, by round $v \geq u + \Delta - 1$, every honest party has adopted a chain of length at least $\ell' = \ell + X'_u + \dots + X'_{v-\Delta}$.*

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Proof. By induction on v .

Basis ($v = u + \Delta - 1$). If at round u an honest party has a chain C of length ℓ , then that party diffuses C at a round earlier than u . It follows that every honest party will receive C by round $u - 1 + \Delta = v$.

Case $X'_{v-\Delta} = 0$. By hypothesis, every honest party has received a chain of length at least $\ell + X'_u + \dots + X'_{v-\Delta-1} = \ell'$ by round $v - 1$.

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Case $X'_{v-\Delta} = 0$. By hypothesis, every honest party has received a chain of length at least $\ell + X'_u + \dots + X'_{v-\Delta-1} = \ell'$ by round $v - 1$.

Case $X'_{v-\Delta} = 1$. By hypothesis, by round $v - \Delta$, every honest party has adopted a chain of length at least

$$\ell + X'_u + \dots + X'_{v-2\Delta} = \ell + X'_u + \dots + X'_{v-\Delta-1} = \ell' - 1.$$

Hence, all honest parties successful at round $v - \Delta$ broadcast a chain of length at least ℓ' . This chain will be received by every honest party by round v .

Concentration for Lipschitz functions

- Note that Y'_i and Y'_j are not independent anymore when $|i - j| < 2\Delta$ and the standard Chernoff bound does not apply. (Similarly for X'_i and X'_j .)

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A function $f(x_1, \dots, x_n)$ is **k -Lipschitz** if $|f(x) - f(x')| \leq k$, whenever x and x' differ in at most one coordinate.

Theorem. *If f is k -Lipschitz and X_1, \dots, X_n are independent random variables, then*

$$\Pr[f > \mathbf{E}f + t] \leq \exp\left(-\frac{2t^2}{nk^2}\right) \quad \text{and} \quad \Pr[f < \mathbf{E}f - t] \leq \exp\left(-\frac{2t^2}{nk^2}\right).$$

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- Each X'_i is a function of $X_{i-\Delta}, \dots, X_i$.
- Thus, the sum $\sum_{i=\Delta}^r X'_i$ is a function of the **independent** random variables X_1, X_2, \dots, X_r .
- Moreover, $\sum_{i=\Delta}^r X'_i$ is **2-Lipschitz** (actually 1-Lipschitz). This is because X_j affects X'_i only if $j \leq i < j + \Delta$ and there can be at most two X'_i equal to 1 in an interval of length Δ .

Chain Quality

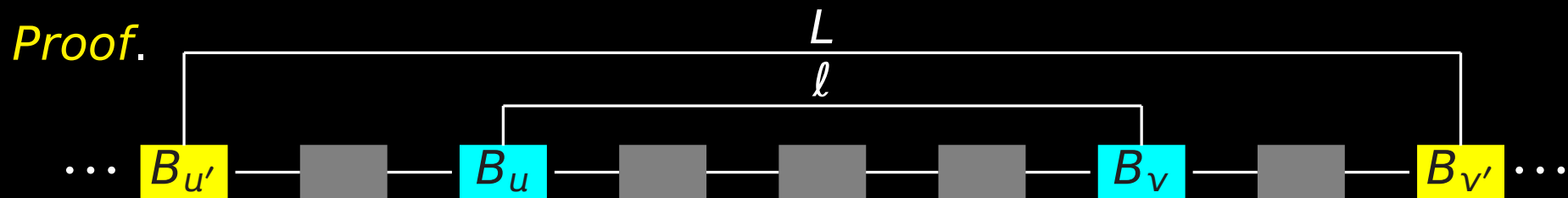
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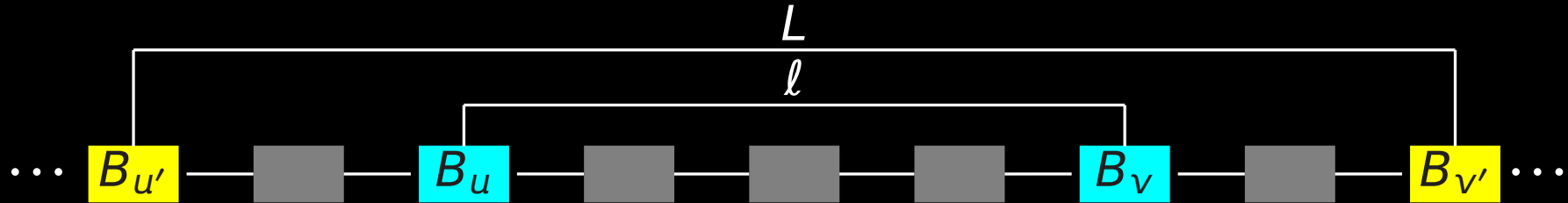
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$$\frac{Z}{L} \leq \frac{Z}{X} \leq \frac{1 + \frac{\epsilon}{3}}{1 - \frac{\epsilon}{3}} \cdot \frac{t}{n} \cdot \left(1 - \frac{\Delta}{|S|}\right) \leq \dots$$

Choose l large enough so that $\Delta/|S|$ is small enough.

Proof of the common-prefix lemma

Common-Prefix Lemma. *The probability that at a given round two parties have chains that disagree in the last k blocks, is at most $e^{-\Omega(k)}$. (The party with the shortest chain should be honest.)*

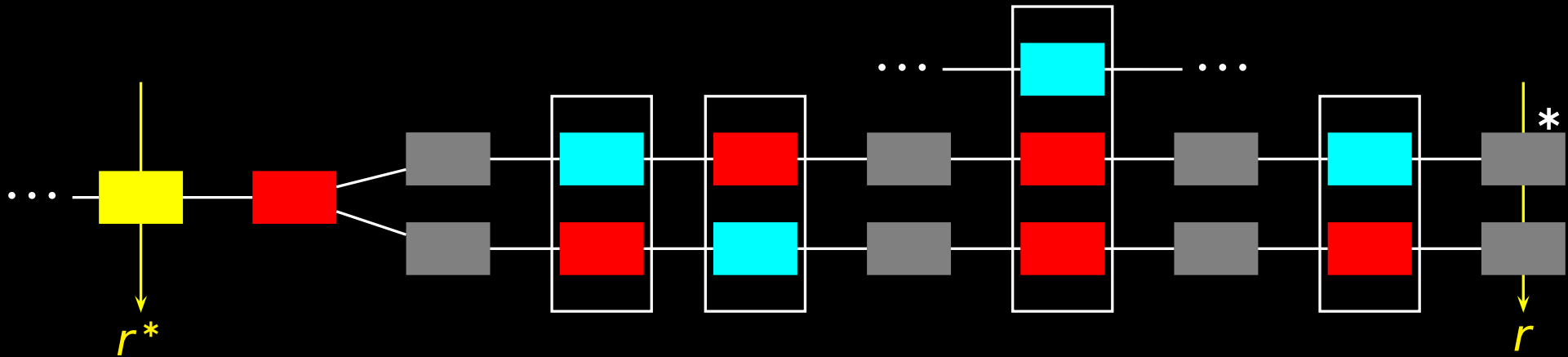
Observation. *Suppose the ℓ -th block of a chain was computed by an honest party in an **isolated uniquely successful round**. Then any other ℓ -th block has been **computed by the adversary**.*

Proof. Suppose a block of height ℓ was computed by an honest party at a round u with $Y'_u = 1$. This implies $X_r = 0$ for and $r \neq u$ with $|r - u| < \Delta$.

- Thus, no honest party could compute another block at a round r with $|r - u| < \Delta$.
- If any honest party computed a block of height ℓ at any round $r < u - \Delta$, then any honest party is trying to extend a chain of length at least ℓ at round u .
- Similarly for $r > u + \Delta$.

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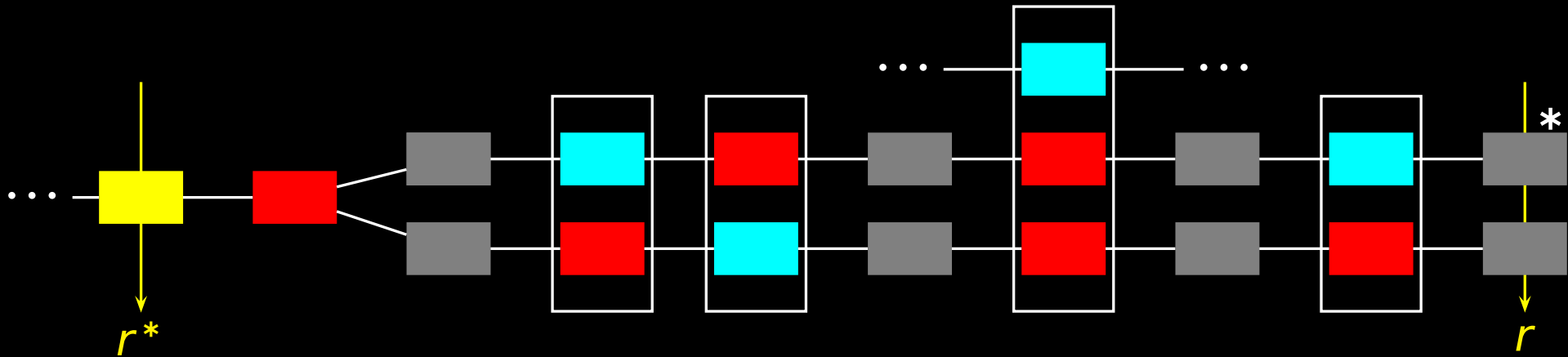


Proof. Let r^* be the last round before the fork that was computed by an honest party. Set

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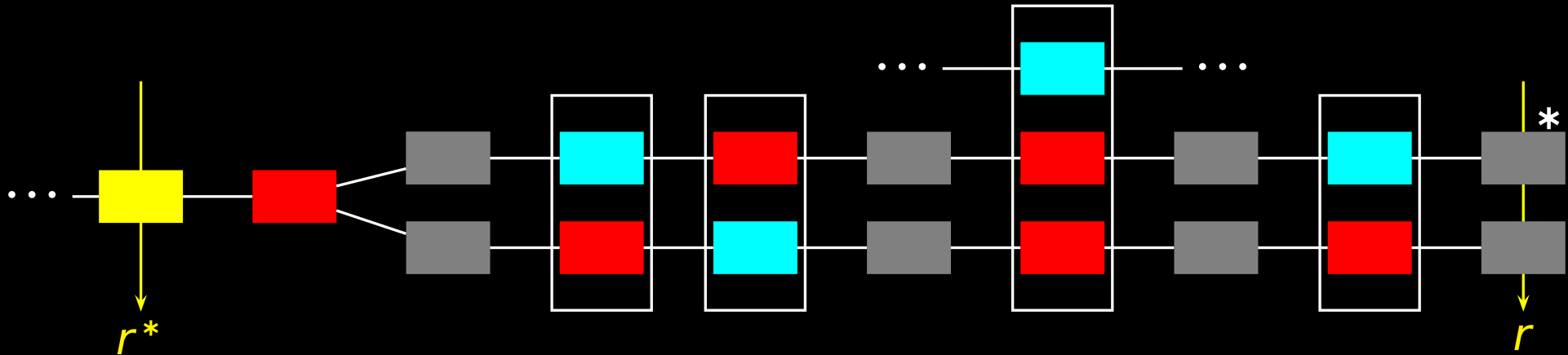
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By the Observation, to every Δ -isolated uniquely successful round in S' **corresponds** an adversarial block computed in S .

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Proof. Let r^* be the last round before the fork that was computed by an honest party. Set

$$S = \{r^* + 1, \dots, r - 1\} \quad \text{and} \quad S' = \{r^* + 1 + \Delta, \dots, r - 1 - \Delta\}.$$

By the Observation, to every Δ -isolated uniquely successful round in S' corresponds an adversarial block computed in S . Thus,

$$\text{Isolated uniquely successful rounds in } S' \leq \text{Adversarial successes in } S.$$

Proof of the common-prefix lemma (cont'd)

Recall $\mathbf{E}[Y'_i] > f(1-f)^{2\Delta-1}$. We can argue that $Y'(S') = \sum_{r \in S'} Y'_r$ is **2-Lipschitz**. By the Concentration bound for Lipschitz functions,

$$\Pr[Y'(S') \leq (1 - \epsilon)f(1-f)^{2\Delta-1}|S'|] = e^{-\Omega(|S|)}.$$

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Assuming these bad events don't occur (union bound) and the Honest Majority Assumption

$$\begin{aligned} Z(S) &< (1+\epsilon)pt|S| \\ &< (1+\epsilon)(1-\delta)pn|S'| && (t < (1-\delta)n) \\ &< (1+\epsilon)(1-\delta) \cdot \frac{f}{1-f} \cdot |S'| \left(1 + \frac{2\Delta}{|S'|}\right) && (1-f)pn < f \\ &\dots \text{(Making } 2\Delta/|S'| \text{ sufficiently small)} \dots \\ &< Y'(S') \end{aligned}$$

Dynamic execution: number of parties is changing

- The proof in the static case (fixed number of parties) breaks.
 - As **block-production rate** goes to **1**, **persistence** breaks.
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- Each block now is associated with a target T and **difficulty $\frac{1}{T}$** .

*Parties now **follow the heaviest chain**.*

Naive target recalculation

- The target is recalculated every m blocks.

Bitcoin uses $m = 2016$ and calls the period between two recalculation points an **epoch**.

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- Suppose the last m blocks were computed in Λ rounds for target T . If we want to have m blocks in every $\frac{m}{f}$ rounds, set

$$T' = \frac{\Lambda}{m/f} \cdot T, \quad (f = \text{block-production rate}).$$

This is justified because for small f the relation between f and T is approximately linear.

Bahack's difficulty raising attack

- Suppose that at some round r the honest parties have a chain of length λm .
- The adversary builds the next epoch all by himself with fake timestamps, resulting in huge difficulty for the next epoch.
- His strategy is to set T' so small, so that if he computes the 1st block (a superblock of difficulty $\frac{1}{T'}$) of the next epoch fast (say half the expected time), he obtains a chain heavier than the chain the honest parties are expected to have by that time.
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But, Nakamoto knew this!!!

Analysis of the attack (sketch)

To see why this works, let us fix a target T for the honest parties and suppose the honest parties advance with success probability f and the adversary with $\frac{1}{1+\delta} \cdot f$ (for some $\delta < 1/2$).

- If the adversary sets $T' = \frac{T}{2\delta m}$, then with constant probability he finishes his attack (i.e., $(m + 1)$ blocks) in

$$(1 + \delta) \cdot \frac{m}{f} + (1 + \delta) \cdot \frac{T}{T'} \cdot \frac{1}{3f}$$

rounds and has collected difficulty

$$\frac{m}{T} + \frac{1}{T'} = \frac{m}{T} + \frac{2\delta m}{T} = (1 + 2\delta) \cdot \frac{m}{T}.$$

- The honest parties have collected (in expectation)

$$(1 + \delta) \left(\frac{m}{T} + \frac{1}{3T'} \right) = (1 + \delta) \left(\frac{m}{T} + \frac{2\delta m}{3T} \right) < (1 + 2\delta) \cdot \frac{m}{T}.$$

The adversary wins with constant probability!

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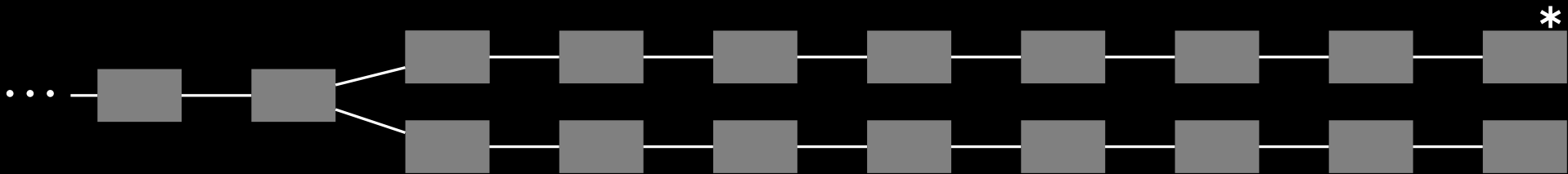
Theorem. *If, for appropriate parameters s and γ ,*

$$\forall r, r' \quad |r - r'| \leq s \implies \frac{n_r}{\lambda} \leq n_{r'} \leq \lambda n_r,$$

then common prefix and chain quality hold (assuming adversarial minority and appropriate initialization).

The common-prefix lemma in the dynamic case

Common-Prefix Lemma. *The probability that at a given round two parties have chains that disagree in the last k blocks, is at most $e^{-\Omega(k)}$. (The party with the shortest chain should be honest.)*



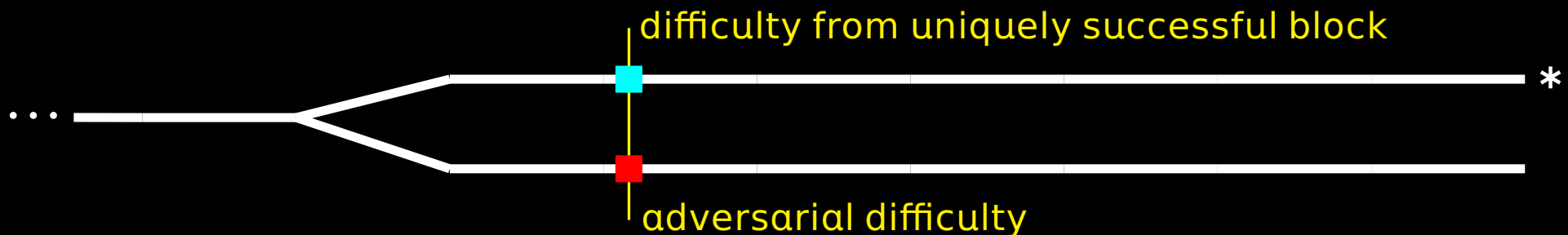
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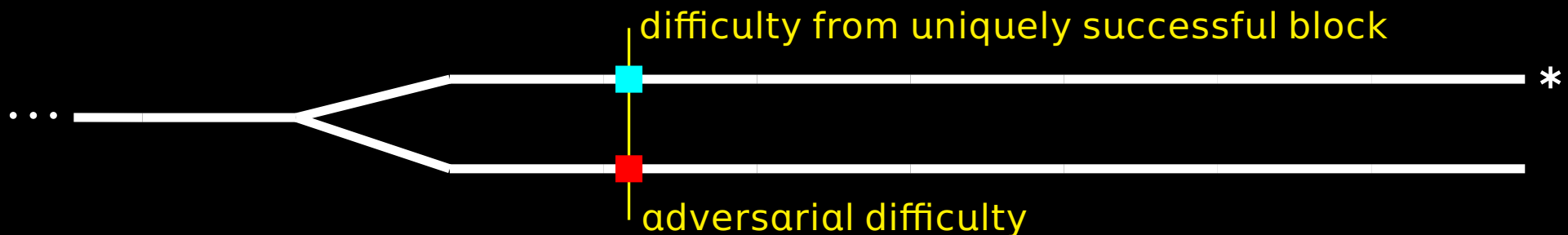
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in uniquely successful
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 - If **heads**, B earns $X_i = \frac{1}{p_i}$ bitcoins; otherwise $X_i = 0$.

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 - If **heads**, B earns $X_i = \frac{1}{p_i}$ bitcoins; otherwise $X_i = 0$.
- **B** is expected to earn $\mathbf{E}[X_i] = 1\text{\$}$ in round i . Thus, **B** is expected to earn $k\text{\$}$ in k rounds.
- How concentrated around their expectation are B's earnings? Does it hold

$$\Pr\left[\sum_{i=1}^k X_i < (1 - \epsilon)k\right] = e^{-\Omega(\epsilon^2 k)} \quad ?$$

Martingale bounds

Theorem. Let f be a function of the n random variables X_1, \dots, X_n .
Let

$$D_i = \mathbf{E}[f|X_1, \dots, X_i] - \mathbf{E}[f|X_1, \dots, X_{i-1}],$$

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(sup is taken over all possible assignments to X_1, \dots, X_{i-1}). Then, for any $t, v \geq 0$,

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Proof application: Show that if an execution begins with good initial parameters (in particular, $V \leq v$) and at some point deviates from the desired block-production rate, then **concentration was violated while $V \leq v$.**

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Theorem. *Every block in a chain that is ever adopted by an honest party, has “accurate” timestamp and “good” target.*

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- Similarly, we shouldn't accept blocks with timestamps too far in the **past**, because **target recalculation** may lead to a small target.
 - Bitcoin considers a block to be valid if its timestamp is at least the **median** of the last **11** timestamps.

Analysis without synchronized clocks

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How honest are the medians?

- If k_{med} is large, then **majority** is not enough for **honest median** (recall selfish mining).
- We need k_{med} to be small enough to argue that honest medians appear **sufficiently often**.

Back in the simplest model

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- Consider the oracle outputs as **votes** for the two candidates: honest and adversary.
- The assumption can be reworded as follows.

We want a **permutation of the votes** so that during the **counting** of the votes the honest candidate **is always ahead** of the adversary.

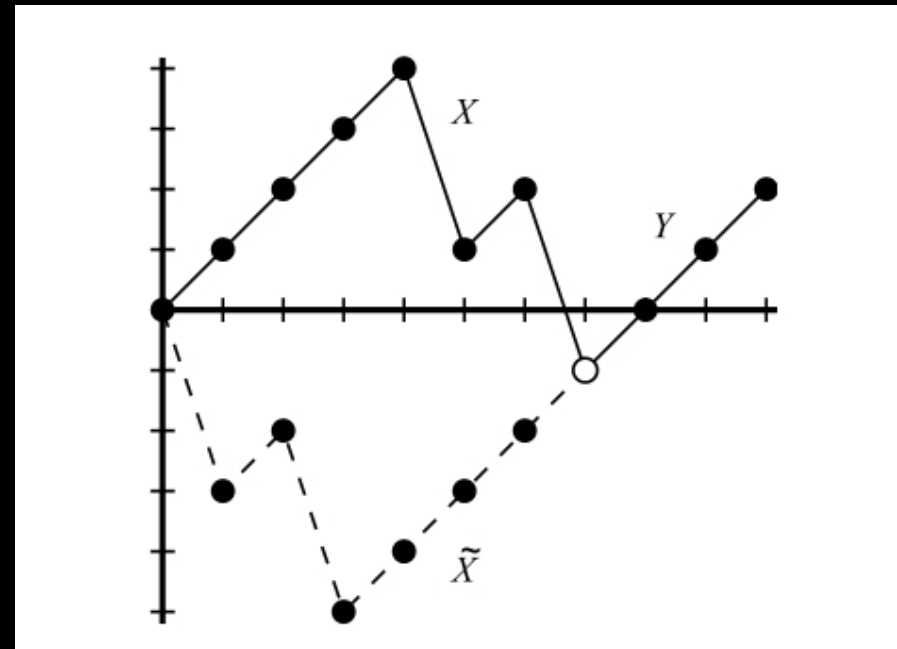
Ballot theorems

Ballot theorem. Suppose candidates A and B received a and b votes respectively. The probability candidate A was **always ahead** during the counting of the votes is

$$\frac{a-b}{a+b}.$$

Proof by reflection.

Four Proofs of the Ballot Theorem,
Marc Renault, Mathematics Magazine,
Vol 80, No 5 (Dec 2007).



Ballot theorems

Theorem. Let X_1, X_2, \dots be an infinite sequence of iid integer random variables with mean $\mu > 0$ and maximum value 1 and for any $i \geq 1$ let $S_i = X_1 + \dots + X_i$. Then

$$\Pr[S_i > 0 \text{ for } n = 1, 2, \dots] = \mu.$$

Addario-Berry and Reed. Ballot Theorems, Old and New. 2008.

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Thank you for listening