Generalized bunched implication logic

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Outline

Motivation:

1. Memory allocation, pointer management, concurrent progamming.
2. Relation algebras (with a view of applications to computer science, program execution, pre and post conditions specification) extended to a constructive setting.
3. Boolean algebras with operators, lattices with operators extended to Heyting algebras with operators.

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1. GBI-algebras: Proof theory and algebra (congruence generation)
2. Weakening relation algebras: a constructive version of relation algebras
3. Do we lose anything from the nice theory of relation algebras?

Separation logic and Bunched Implication Logic

Given an (infinite) set L of *locations* and a set RV of *record values*, a *heap* is a finite partial function from L to RV. For two heaps h, h', we define $h \cdot h'$ to be the union, when their domains are disjoint and undefined otherwise. The set H of heaps is partially ordered (by inclusion) partial monoid.

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It has a nice proof theory (an analytic calculus with cut elimination) and we generalize it to the non-commutative setting.

The Gentzen calculus for GBI

$$\frac{x \Rightarrow a \quad u(a) \Rightarrow c}{u(x) \Rightarrow c}\ (\mathsf{CUT}) \qquad \frac{}{a \Rightarrow a}\ (\mathsf{Id}) \qquad \frac{u(x \owedge (y \owedge z)) \Rightarrow c}{u((x \owedge y) \owedge z) \Rightarrow c}\ (\owedge a)$$

$$\frac{u(x \owedge y) \Rightarrow c}{u(y \owedge x) \Rightarrow c}\ (\owedge e) \qquad \frac{u(x) \Rightarrow c}{u(x \owedge y) \Rightarrow c}\ (\owedge i) \qquad \frac{u(x \owedge x) \Rightarrow c}{u(x) \Rightarrow c}\ (\owedge c)$$

$$\frac{u(a) \Rightarrow c \quad u(b) \Rightarrow c}{u(a \vee b) \Rightarrow c}\ (\vee\mathsf{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee\mathsf{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee\mathsf{R}r)$$

$$\frac{u(a \owedge b) \Rightarrow c}{u(a \wedge b) \Rightarrow c}\ (\wedge\mathsf{L}) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge\mathsf{R})$$

$$\frac{u(a \circ b) \Rightarrow c}{u(a \cdot b) \Rightarrow c}\ (\cdot\mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot\mathsf{R}) \qquad \frac{u(\varepsilon) \Rightarrow a}{u(1) \Rightarrow a}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathsf{R})$$

$$\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u(x \circ (a \backslash b)) \Rightarrow c}\ (\backslash\mathsf{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash\mathsf{R}) \qquad \frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u((b/a) \circ x) \Rightarrow c}\ (/\mathsf{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathsf{R})$$

$$\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u(x \owedge (a \to b)) \Rightarrow c}\ (\to\mathsf{L}) \qquad \frac{x \owedge a \Rightarrow b}{x \Rightarrow a \to b}\ (\to\mathsf{R}) \qquad \frac{u(\delta) \Rightarrow c}{u(\top) \Rightarrow c}\ (\top\mathsf{L}) \qquad \frac{}{x \Rightarrow \top}\ (\top\mathsf{R})$$

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

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Heying RL's

To allow for negation in the language we consider expansions of residuated lattices with an additional *negation constant* 0 and we define two (*linear*) negations

$$\sim x = x \backslash 0 \text{ and } -x = 0/x.$$

An *involutive residuated lattice* is such an expansion that satisfies *double negation*: $\sim - x = x = - \sim x$.

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It is called *involutive* if the dynamic part is involutive. It is called *Boolean* if the logical part is involutive ($\neg\neg x = x$).

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Theorem. Relation algebras are precisely the Boolean InRL's that satisfy $\neg{\sim}(xy) = \neg{\sim}y \cdot \neg{\sim}x$. (They also satisfy ${\sim}x = -x$ and ${\sim}\neg x = \neg{\sim}x$.)

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[Tarski, Givant] The equational theory of relation algebras (even with ground terms only) can serve as a foundations for most of mathematics (number theory and set theory, included). RA can interpret the 3-variable fragment of first-order logic.

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We will consider a non-Boolean generalization of these relation algebras that can be thought of as intuitionistic relation algebras.

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2. determining subdirectly irreducibles algebras
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The maps $\theta \mapsto \uparrow([1]_\theta)$ and $M \mapsto \theta_M$ define a lattice isomorphism, where

$$a \; \theta_M \; b \text{ iff } a\backslash b, b\backslash a \in M.$$

Congruences in Heyting RL: $\uparrow[1]$ and $[\top]$

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Corollary. Heyting InRLs enjoy a *local* deduction theorem and have the *Congruence Extension Property*.

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- $R \wedge S = R \cap S$
- $R \vee S = R \cup S$
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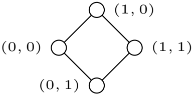
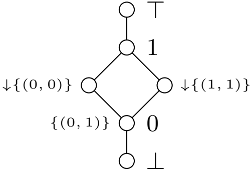
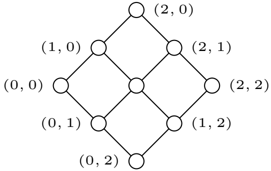
Weakening relations on 2



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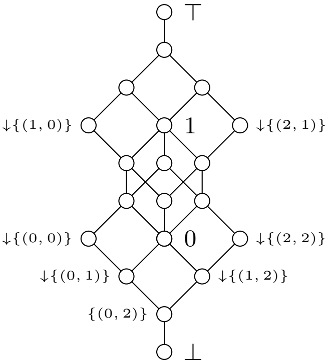
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If $\mathcal{K}$ is a class of Heyting RLs, we define the class of all double-division conuclei images of algebras in $\mathcal{K}$ by positive idempotents:

$$\mathsf{d}\mathcal{K} = \{\mathbf{A}_\mathsf{p} : \mathbf{A} \in \mathcal{K}, 1 \leq \mathsf{p} = \mathsf{p}^2, \mathsf{p} \in \mathsf{A}\}$$

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Question: By passing to weakening algebras we gain a constructive aspect, but do we lose any of these nice structural algebraic properties?

Answer: We do not lose any of these nice properties! ☺

If $\mathcal{K}$ is a class of Heyting RLs, we define the class of all double-division conuclei images of algebras in $\mathcal{K}$ by positive idempotents:

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We define *representable weakening relation algebras* as: $\mathsf{RWkRA} = \mathsf{ISP}(\mathsf{FWkRA})$.
Question: Is RWkRA is a variety? Is it a nice variety?

Discriminator varieties

Lemma. Let $\mathcal{K}$ be a class of residuated lattices or Heyting RLs. Then

1. $\mathsf{dP}\mathcal{K} = \mathsf{Pd}\mathcal{K}$.
2. $\mathsf{dS}\mathcal{K} \subseteq \mathsf{Sd}\mathcal{K}$.
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If the SI algebras in a variety have a *discriminator term* $t(x, y, z)$:

$$t(x, y, z) = z \text{ if } x = y \text{ and } t(x, y, z) = x \text{ if } x \neq y.$$

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Fact. In SI relation algebras, $\top \backslash x / \top$ is a unary discriminator term.

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Corollary. The class SdRRA is a discriminator variety with unary discriminator term $\top\backslash x/\top$. Its class of simple members is exactly SdSi(RRA).

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Thank you!! ☺