Categorifying Borel Reducibility

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Any reasonably hands-on mathematical construction is Borel.

A Polish space where you forget the topology and just remember which sets are Borel is called a *standard Borel space*.

Borel reducibility

Definition (Friedman and Stanley, 1989)

Let

- X and Y be standard Borel spaces,
- E be an equivalence relation on X, and
- F be an equivalence relation on Y.

We say that E is *Borel reducible* to F, and write $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that

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If you can show that there is *no* such Borel reduction, the classification programme cannot be completed. This has led to important unclassifiability results in, e.g.,

- ergodic theory (Hjorth, and Foreman & Weiss)
- C^*-algebras (Farah, Toms & Törnquist).

Example

Let $\mathcal{V}$ be a countable vocabulary, with

- relation symbols R_m of arity a_m and
- function symbols f_n of arity b_n.

We consider countable $\mathcal{V}$-structures. We might as well assume they all have underlying set $\mathbb{N}$.

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Each such structure can by encoded by a single point in

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Similarly, for any first order (or $\mathcal{L}_{\omega_1,\omega}$) sentence φ over $\mathcal{V}$, we can consider the subspace $\operatorname{Mod}(\varphi)$ of $\operatorname{Str}(\mathcal{V})$ of models of φ.

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The isomorphism relation of a class of structures $\mathrm{Mod}(\varphi)$ is *Borel complete* if every isomorphism relation on a class $\mathrm{Mod}(\psi)$ of structures Borel reduces to it.

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Examples

- Graphs
- Groups (Mekler)
- Rooted trees (Friedman & Stanley)
- Linear orders (Friedman & Stanley)
- Fields (Friedman & Stanley)
- Boolean algebras (Camerlo & Gao)
- Quandles (B.-T. & Miller)

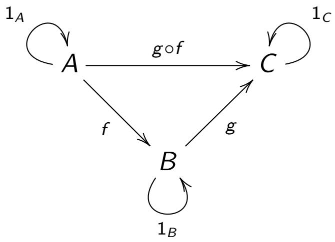
Categories

Definition

A *category* $\mathcal{C}$ consists of

- a class $\mathrm{Obj}\,\mathcal{C}$ of *objects*, and
- a class $\mathrm{Mor}\,\mathcal{C}$ of *morphisms* between objects,

with a reasonable notion of composition of morphisms, and identity morphisms.



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A *category* $\mathcal{C}$ consists of

- a class $\operatorname{Obj}\mathcal{C}$ of *objects*, and
- a class $\operatorname{Mor}\mathcal{C}$ of *morphisms*,
- functions $\operatorname{dom}, \operatorname{cod} \colon \operatorname{Mor}\mathcal{C} \to \operatorname{Obj}\mathcal{C}$,
- an identity morphism function $\mathrm{id} \colon \operatorname{Obj}\mathcal{C} \to \operatorname{Mor}\mathcal{C}$,
- a composition partial binary function $\circ$ on morphisms, giving a morphism for each pair (g, f) of morphisms such that $\operatorname{dom}(g) = \operatorname{cod}(f)$,

satisfying the following axioms:[a]

- When they are defined, $(h \circ g) \circ f = h \circ (g \circ f)$.
- $\forall a \in \operatorname{Obj}\mathcal{C}$, $\operatorname{dom}(\mathrm{id}(a)) = \operatorname{cod}(\mathrm{id}(a)) = a$.
- $\forall f \in \operatorname{Mor}\mathcal{C}$, $\mathrm{id}(\operatorname{cod}(f)) \circ f = f \circ \mathrm{id}(\operatorname{dom}(f)) = f$.

[a] Also, $\operatorname{dom}^{-1}(A) \cap \operatorname{cod}^{-1}(B) = \operatorname{Hom}(A, B)$ is usually required to be a *set* for all $A, B \in \operatorname{Obj}\mathcal{C}$.

E.g.

For any vocabulary $\mathcal{V}$ and any $\mathcal{V}$-sentence φ, we have the category whose objects are all models of φ, and whose morphisms are the $\mathcal{V}$-homomorphisms. Such categories are called *algebraic*.

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For any vocabulary $\mathcal{V}$ and any $\mathcal{V}$-sentence φ, we have the category whose objects are all models of φ, and whose morphisms are the $\mathcal{V}$-homomorphisms. Such categories are called *algebraic*.

A functor is a homomorphism of categories: it takes objects to objects and morphisms to morphisms in a way that respects composition, identities, domains and codomains.

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Idea: extend the Borel reducibility framework to accommodate this.

Borel categories

Definition

A *Borel category* $\mathcal{C}$ is a pair $(\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$ of standard Borel spaces endowed with

- Borel functions dom and cod from $\mathrm{Mor}(\mathcal{C})$ to $\mathrm{Obj}(\mathcal{C})$,
- a Borel function id from $\mathrm{Obj}(\mathcal{C})$ to $\mathrm{Mor}(\mathcal{C})$, and
- a Borel function $\circ$ from $\{(g, f) \in \mathrm{Mor}(\mathcal{C})^2 \mid \mathrm{dom}(g) = \mathrm{cod}(f)\}$ to $\mathrm{Mor}(\mathcal{C})$,

such that $(\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$ with these functions is a category.

Examples

Let $\operatorname{Mor}\mathcal{V}$ be the subset of $\operatorname{Str}\mathcal{V} \times \operatorname{Str}\mathcal{V} \times \mathbb{N}^{\mathbb{N}}$ consisting of those $(\mathcal{M}, \mathcal{N}, f)$ such that f is a $\mathcal{V}$-homomorphism from $\mathcal{M} \to \mathcal{N}$. Then $\mathbf{Str}\,\mathcal{V} = (\operatorname{Str}\mathcal{V}, \operatorname{Mor}\mathcal{V})$ is a Borel category.

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For any sentence φ over $\mathcal{V}$,

$$\mathbf{Mod}(\varphi) = (\operatorname{Mod}\varphi, \operatorname{Mor}\mathcal{V} \cap (\operatorname{Mod}\varphi \times \operatorname{Mod}\varphi \times \mathbb{N}^{\mathbb{N}}))$$

is a Borel category.

Borel functors

Definition

A *Borel functor* from a Borel category $\mathcal{C}$ to a Borel category $\mathcal{D}$ is a pair of Borel functions $F_{\mathsf{Obj}} \colon \mathsf{Obj}(\mathcal{C}) \to \mathsf{Obj}(\mathcal{D})$ and $F_{\mathsf{Mor}} \colon \mathsf{Mor}(\mathcal{C}) \to \mathsf{Mor}(\mathcal{D})$ that respect the functions dom, cod, id and $\circ$.

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We generally drop the subscripts writing $F(A)$ and $F(f)$.

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Note

Borel functors must respect isomorphisms: if $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$ then

$$F(g) \circ F(f) = F(\mathrm{id}) = \mathrm{id} \text{ and } F(f) \circ F(g) = F(\mathrm{id}) = \mathrm{id}.$$

But they need not respect non-isomorphism.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *full* if for all objects C_0 and C_1 of $\mathcal{C}$, F restricted to $\mathrm{Hom}_{\mathcal{C}}(C_0, C_1)$ is surjective:

$$\forall g\colon F(C_0) \to F(C_1)\, \exists f\colon C_0 \to C_1\, (F(f) = g).$$

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A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *full embedding* if it is full and injective both on objects and morphisms.

Note that a fully faithful F can only fail to be an embedding by identifying isomorphic objects.

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Proposition

The object map F_{Obj} of a functorial reduction $F\colon \mathcal{C} \to \mathcal{D}$ is a Borel reduction from the isomorphism relation on $\mathrm{Obj}(\mathcal{C})$ to the isomorphism relation on $\mathrm{Obj}(\mathcal{D})$.

Proof

It suffices to show that if $F(C_0) \cong_{\mathcal{D}} F(C_1)$, then $C_0 \cong_{\mathcal{C}} C_1$.

Functorial reductions are Borel reductions

If $F(C_0) \cong_{\mathcal{D}} F(C_1)$, that means there are morphisms $h\colon F(C_0) \to F(C_1)$ and $j\colon F(C_1) \to F(C_0)$ such that

$$j \circ h = \mathrm{id}(F(C_0)) \qquad \text{and} \qquad h \circ j = \mathrm{id}(F(C_1)).$$

By fullness there are $f\colon C_0 \to C_1$ and $g\colon C_1 \to C_0$ such that $F(f) = h$ and $F(g) = j$. By functoriality,

$$F(g \circ f) = F(g) \circ F(f) = j \circ h = \mathrm{id}(F(C_0)) = F(\mathrm{id}(C_0)), \quad \text{and}$$
$$F(f \circ g) = F(f) \circ F(g) = h \circ j = \mathrm{id}(F(C_1)) = F(\mathrm{id}(C_1)),$$

and then by faithfulness,

$$g \circ f = \mathrm{id}(C_0) \qquad \textit{and} \qquad f \circ g = \mathrm{id}(C_1).$$

So $C_0 \cong C_1$. □

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How does this compare with the category theory literature?

There is a substantial body of work that has been done on the analogous category theory notions (without "Borel").

- "Hands-on" $\longrightarrow$ probably translate to Borel setting.
- Care required to keep the constructions countable.

Categorifying Borel completeness

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A Borel category of the form **Mod**(φ) is *functorially universal* if every Borel category **Mod**(ψ) functorially reduces to it.

Which Borel complete classes of structures become functorially universal Borel categories?

Graphs

Let **Gra** be the Borel category of directed graphs on $\mathbb{N}$, with directed graph homomorphisms as morphisms. Equivalently, this is $\mathbf{Str}\{E\}$ where E is a binary relation.

Proposition

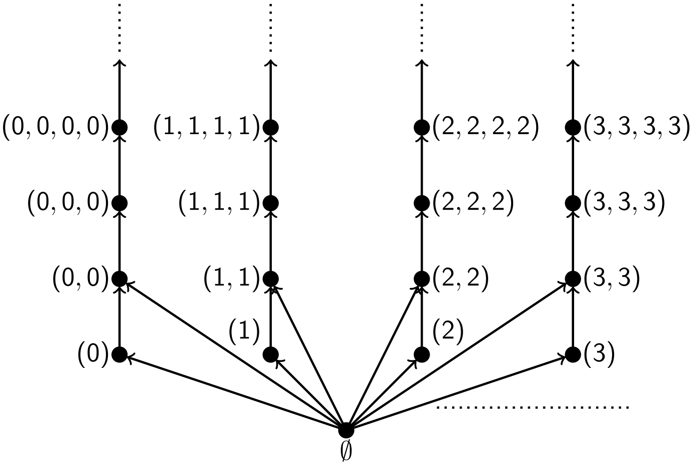
Gra *is functorially universal.*

There are various proofs of similar statements in the literature, with proofs that don't quite carry across. E.g.:

- The proof in Hodges (Theorem 5.5.1) in the context of interpretability is functorial, but not full.
- A proof from the category theory literature (Pultr and Trnkova Section II.5) doesn't preserve countability.

But adjustments can be found to make them work.

T:



Having this relation in your structure allows you to encode finite tuples from $\mathbb{N}$ into $\mathbb{N}$, in such a way that respecting the relation ensures you respect the length of the encoded tuples.

Mono-unary algebras

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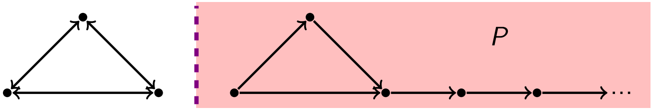
Proof sketch

Hedrlín and Pultr showed that if the endomorphism monoid of a mono-unary algebra is a group, then it is a direct product of cyclic groups. So for example, $\mathbf{Str}\{f\}$ does not admit a functorial reduction from $\mathbf{Str}\{E, P\}$ where E is a binary relation and P is a unary relation, since the following structure has endomorphism mondoid S_3:



Di-unary algebras

Let f and g be unary function symbols. Then

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Proof Sketch (essentially Hedrlín and Pultr)

Embed **Gra** in **Str**$\{f, g\}$ by taking (V, E) to an $\{f, g\}$-structure with underlying set $V \cup E \cup 2$, where if $e \in E$ is an edge from v_0 going to v_1, we define $f(e) = v_0$ and $g(e) = v_1$ (and define f and g on V and 2 in a suitable way that the only homomorphisms are those arising from graph homomorphisms).

Linear Orders

Let **LinOrd** be the Borel category of linear orders with underlying set $\mathbb{N}$, with order-preserving functions as homomorphisms. Friedman and Stanley showed that the isomorphism relation of countable linear orders is Borel complete. But:

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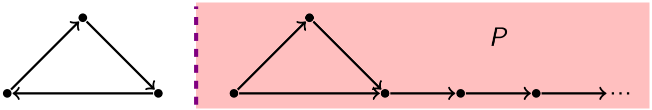
Proof sketch

Every non-trivial automorphism of a linear order has infinite order in the automorphism group. So for example, $\mathbf{Str}\{f\}$ does not admit a functorial reduction from $\mathbf{Str}\{E, P\}$ where E is a binary relation and P is a unary relation, since this structure has automorphism group $\mathbb{Z}/3\mathbb{Z}$:



Boolean algebras

Let **BA** be the Borel category of Boolean algebras with underlying set $\mathbb{N}$, with Boolean algebra homomorphisms as the morphisms. Camerlo and Gao showed that the isomorphism relation of countable Boolean algebras is Borel complete. But:

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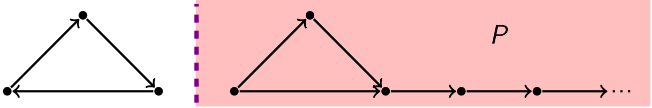
Proof sketch

McKenzie and Monk showed that every element of the centre of the automorphism group of any Boolean algebra has order at most 2. So for example, $\mathbf{Str}\{f\}$ does not admit a functorial reduction from $\mathbf{Str}\{E, P\}$ where E is a binary relation and P is a unary relation, since this structure has automorphism group $\mathbb{Z}/3\mathbb{Z}$:



Questions

Hjorth's notion of *turbulence* is a central tool for the impossibility of classification results mentioned at the start — can we find a categorified version of that?

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Can we use this framework to show some classification programme is impossible, where standard Borel reducibility was too blunt a tool?

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