Precise expressions for the algorithmic information distance

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I am also grateful to
– Mikhail Andreev,
– Artem Grachev,
– the participants of the Kolmogorov seminar in Moscow,
for useful discussions.

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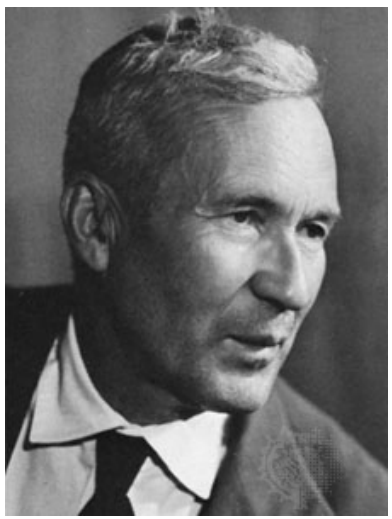

A. Kolmogorov

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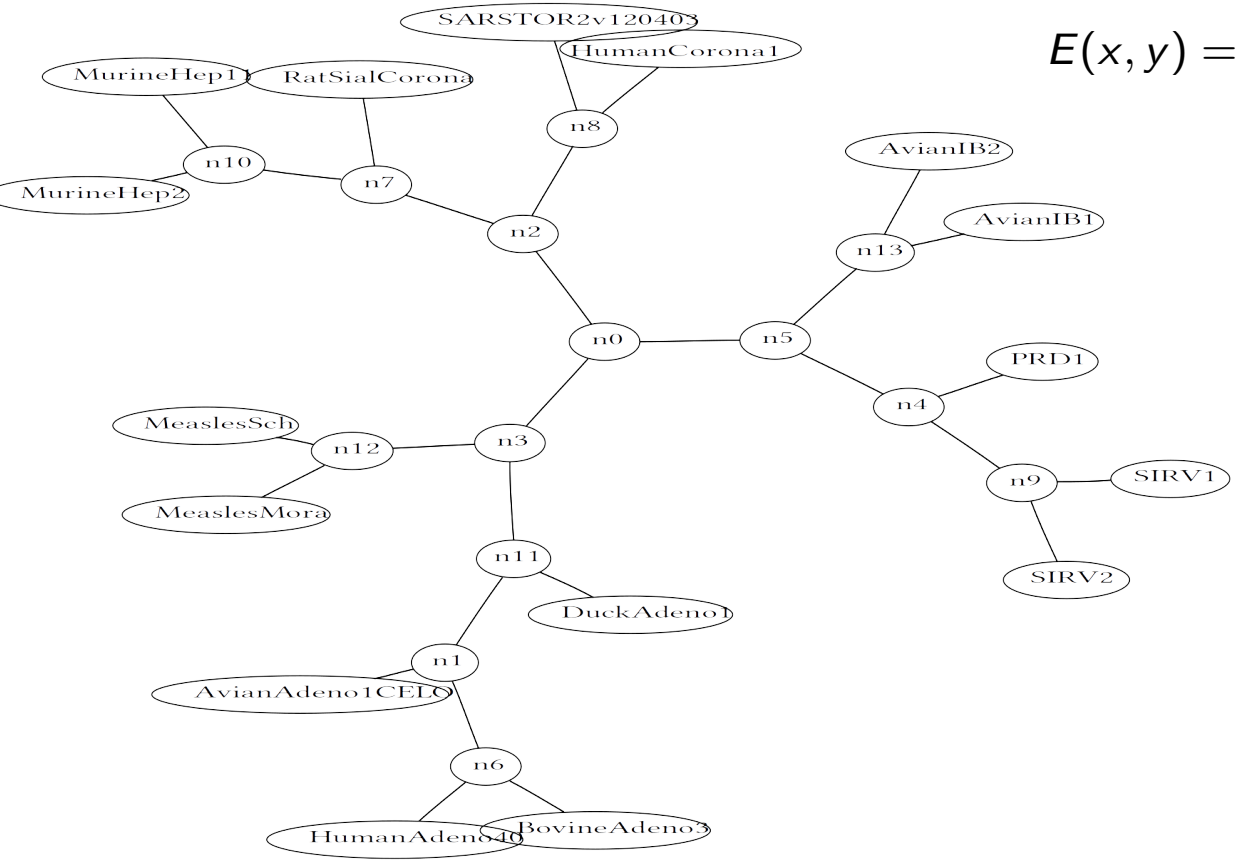
  (ii) has balls $\{y : D(x, y) \leq k\}$ of size at most 2^k.
- Characterization: $E(x, y) = \max\{\mathrm{C}(x|y), \mathrm{C}(y|x)\} + O(\log n)$.
  Applications: approximate complexity with real compressors.

Applications [Cilibrasi and Vitanyi, 2005]

Below, some SARS-related viruses (including some Coronaviruses from 2003).



$$E(x, y) = \max\{\mathrm{C}(x|y), \mathrm{C}(y|x)\} + O(\log n).$$

When the approximated distance is applied to tree-cluster algorithm, it can recover non-obvious information:

- Group music files by genre.
- Group Russian books by authors.
- Recover phylogenetic trees in biology.
- ...

Disclaimer: a normalized variant of the distance was approximated.

15 virus genomes were downloaded from The Universal Virus Database of the International Committee on Taxonomy of Viruses and Canadas Michael Smith Genome Sciences Centre when they were sequenced. The algorithm recovered the evolutionary tree as was published in the in the New England Journal of Medicine in April 2003. (Submitted to ArXiv in Dec 2003.)

Outline

2 variants of the distance:
with plain and prefix complexity

	plain	prefix
triangle inequality	$\times$	?
$=$ max complexity	$\checkmark$	?

The plain information distance

$$E_U(x,y) = \min\{|p|\colon U(p,x) = y \text{ and } U(p,y) = x\},$$

We use machines U that minimize this distance up to $O(1)$ and drop index U.

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We construct a machine U for which $U(p,x) = y$ and $U(p,y) = x$ if $(x,y) \in R_k$. □

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- This work: we prove that it is false, but holds if $|x| = |y|$ and $E(x, y) \geq 6 \log |x|$.

Main results

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If $|x| = |y|$ *and* $E'(x, y) \geq 6 \log |x|$, *then* $E(x, y) = E'(x, y) + O(1)$.

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Open questions:

- Does E satisfy the triangle inequality up to $O(1)$?
  Recall that E' satisfies the triangle inequality.
- We can define the information distance with
  - 2 machines U and V: min $|p|$ such that $U(p,x) = y$ and $V(p,y) = x$,
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Questions?

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