Theorem Proving in Haskell

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Theorem Proving in Haskell

e.g., theorem:

$$\phi \equiv (\exists x. \forall y. (p\ x\ y)) \Rightarrow (\forall y. \exists x. (p\ x\ y))$$

Encode Theorem as Haskell Type,

then Proof is a Haskell Expression.

Theorem Proving in Haskell

I. Refinement Types

II. Theorems as Types

III. Proofs as Programs

I. Refinement Types

42 :: Int

42 :: {v:Int | v == 42}

Base type

Refinement

```
2 :: {v:Int | v == 2}
42 :: {v:Int | v == 42}
div :: Int -> {v:Int | v /= 0} -> Int
```

div 42 2 :: ??

2 :: {v:Int | v == 2}

42 :: {v:Int | v == 42}

div :: Int -> {v:Int | v /= 0} -> Int

$$\frac{42 :: \mathrm{Int} \qquad 2 :: \{v{:}\mathrm{Int} \mid v \mathrel{/=} 0\}}{\mathrm{div}\ 42\ 2 :: ??}$$

2 :: {v:Int | v == 2}

42 :: {v:Int | v == 42}

div :: Int -> {v:Int | v /= 0} -> Int

We know

We want

$$\frac{2 :: \{v{:}Int \mid v == 2\} \Rightarrow 2 :: \{v{:}Int \mid v \mathrel{/=} 0\}}{2 :: \{v{:}Int \mid v \mathrel{/=} 0\}}$$

That is subtyping!

t_1 <: t_2 iff. $\forall$e. e::t_1 $\Rightarrow$ e::t_2

$$\frac{\{v{:}Int \mid v == 2\} <: \{v{:}Int \mid v \mathrel{/}= 0\}}{2 :: \{v{:}Int \mid v == 2\} \Rightarrow 2 :: \{v{:}Int \mid v \mathrel{/}= 0\}}$$

$$\frac{}{\{v{:}\mathrm{Int} \mid v == 2\} <: \{v{:}\mathrm{Int} \mid v \;/\!= 0\}}$$

Implication implies subtyping

$$\frac{\forall\ v.\ v == 2 \Rightarrow v\ /= 0}{\{v{:}Int\ \mid\ v == 2\} <: \{v{:}Int \mid v\ /= 0\}}$$

Putting it all together

$$\dfrac{\dfrac{42 :: \text{Int} \qquad \dfrac{2 :: \{v{:}\text{Int} \mid v == 2\} \quad \dfrac{\forall v.\ v == 2 \Rightarrow v \mathrel{/}= 0}{\{v{:}\text{Int} \mid v == 2\} <: \{v{:}\text{Int} \mid v \mathrel{/}= 0\}}}{2 :: \{v{:}\text{Int} \mid v \mathrel{/}= 0\}}}{}}{\text{div } 42\ 2 :: \text{Int}}$$

Specifications

div :: Int -> {v:Int | v /= 0} -> Int

42 :: {v:Int | v == 42}

2 :: {v:Int | v == 2}

Observe: Implications have the language of specifications

$$\dfrac{42 :: \mathrm{Int} \qquad \dfrac{2 :: \{v:\mathrm{Int} \mid \boxed{v == 2}\} \qquad \dfrac{\forall\, v.\ \boxed{v == 2} \Rightarrow \boxed{v \mathbin{/=} 0}}{\{v:\mathrm{Int} \mid \boxed{v == 2}\} <: \{v:\mathrm{Int} \mid \boxed{v \mathbin{/=} 0}\}}}{2 :: \{v:\mathrm{Int} \mid \boxed{v \mathbin{/=} 0}\}}}{\mathrm{div}\ 42\ 2 :: \mathrm{Int}}$$

Specifications

$$\mathrm{div} :: \mathrm{Int} \to \{v:\mathrm{Int} \mid \boxed{v \mathbin{/=} 0}\} \to \mathrm{Int}$$

$$42 :: \{v:\mathrm{Int} \mid v == 42\}$$

$$2 :: \{v:\mathrm{Int} \mid \boxed{v == 2}\}$$

Let p be the language of refinements

Implications

$\forall \overline{x} . p \Rightarrow p$

$$\dfrac{42 :: \text{Int} \qquad \dfrac{2 :: \{v{:}\text{Int} \mid \boxed{v == 2}\} \quad \dfrac{\forall v. \boxed{v == 2} \Rightarrow \boxed{v \mathrel{/=} 0}}{\{v{:}\text{Int} \mid \boxed{v == 2}\} <: \{v{:}\text{Int} \mid \boxed{v \mathrel{/=} 0}\}}}{2 :: \{v{:}\text{Int} \mid \boxed{v \mathrel{/=} 0}\}}}{\text{div } 42 \ 2 :: \text{Int}}$$

Specifications

over p

$\text{div} :: \text{Int} \rightarrow \{v{:}\text{Int} \mid \boxed{v \mathrel{/=} 0}\} \rightarrow \text{Int}$

$42 :: \{v{:}\text{Int} \mid v == 42\}$

$2 :: \{v{:}\text{Int} \mid \boxed{v == 2}\}$

Let p be the language of refinements

Implications

$\forall \overline{x} \,.\, p \Rightarrow p$

$$\dfrac{42 :: \mathrm{Int} \qquad \dfrac{\dfrac{\forall v.\ \boxed{v == 2} \Rightarrow \boxed{v \mathrel{/=} 0}}{2 :: \{v{:}\mathrm{Int} \mid \boxed{v == 2}\} \quad \{v{:}\mathrm{Int} \mid \boxed{v == 2}\} <: \{v{:}\mathrm{Int} \mid \boxed{v \mathrel{/=} 0}\}}}{2 :: \{v{:}\mathrm{Int} \mid \boxed{v \mathrel{/=} 0}\}}}{\boxed{\Gamma \vdash e :: t}}$$

Specifications

over p

div :: Int -> {v:Int | v /= 0} -> Int

42 :: {v:Int | v == 42}

2 :: {v:Int | v == 2}

Let p be the language of refinements

Implications

$\forall\ \overline{x}\ .\ p \Rightarrow p$

$$\forall\ v.\ v == 2 \Rightarrow v \mathbin{/=} 0$$

$$\dfrac{\forall\ v.\ v == 2 \Rightarrow v \mathbin{/=} 0}{2 :: \{v{:}Int \mid v == 2\}\quad \{v{:}Int \mid v == 2\} <: \{v{:}Int \mid v \mathbin{/=} 0\}}$$

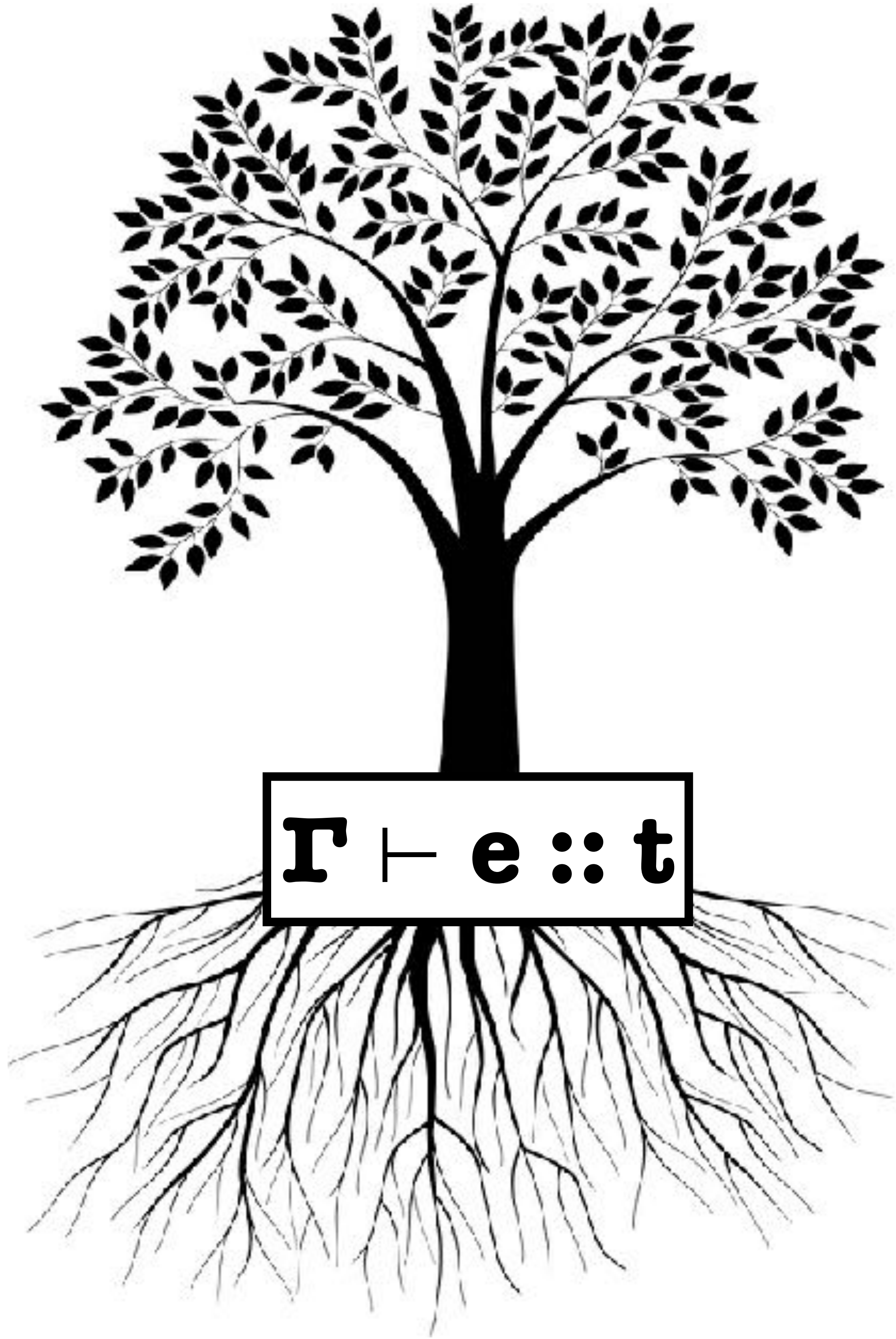
$$\dfrac{42 :: Int \qquad\qquad 2 :: \{v{:}Int \mid v \mathbin{/=} 0\}}{\boxed{\boldsymbol{\Gamma \vdash e :: t}}}$$

Specifications

over p

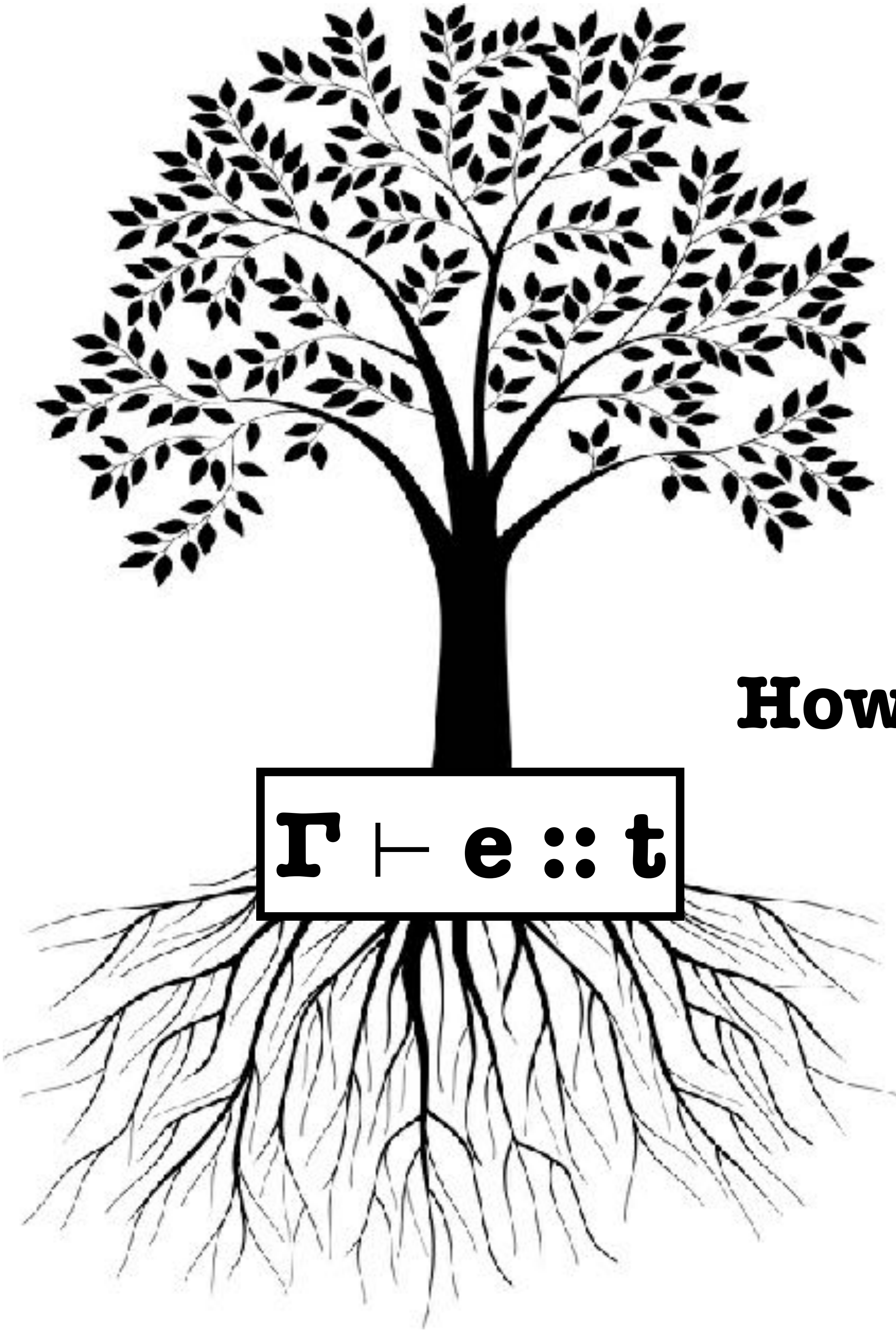
Implications

$\forall\, \overline{x}\,.\, p \Rightarrow p$



Specifications

over p



Implications

$\forall \overline{x} . p \Rightarrow p$

For decidable implications,
p should be quantifier free.

How to express quantified theorems?

Specifications

over p

Theorem Proving in Haskell

I. Refinement Types

II. Theorems as Types

II. Theorems as Types
(a.k.a. Curry Howard Correspondence)

	Logical Formula
Native Term	e
Conjunction	$\varphi_1 \wedge \varphi_2$
Disjunction	$\varphi_1 \vee \varphi_2$
Implication	$\varphi_1 \Rightarrow \varphi_2$
Forall	$\forall\ x.\ \varphi$
Exists	$\exists\ x.\ \varphi$

	Logical Formula	Refinement Type
Native Term	e	
Conjunction	$\varphi_1 \land \varphi_2$	
Disjunction	$\varphi_1 \lor \varphi_2$	
Implication	$\varphi_1 \Rightarrow \varphi_2$	
Forall	$\forall\ x.\ \varphi$	
Exists	$\exists\ x.\ \varphi$	

	Logical Formula	Refinement Type
Native Term	e	??
Conjunction	$\varphi_1 \land \varphi_2$	
Disjunction	$\varphi_1 \lor \varphi_2$	
Implication	$\varphi_1 \Rightarrow \varphi_2$	
Forall	$\forall x.\ \varphi$	
Exists	$\exists x.\ \varphi$	

Native Term e as a Refinement Type

$$\{v{:}() \mid e\}$$

Some unit value

so that e is true.

	Logical Formula	**Refinement Type**
Native Term	e	{v:() | e }
e.g.,	42 <= 42	{v:() | 42 <= 42 }
	x == y	{v:() | x == y }
	p x y	{v:() | p x y }

	Logical Formula	Refinement Type
Native Term	e	$\{v:() \mid e\}$
Conjunction	$\varphi_1 \wedge \varphi_2$	??
Disjunction	$\varphi_1 \vee \varphi_2$	
Implication	$\varphi_1 \Rightarrow \varphi_2$	
Forall	$\forall x.\ \varphi$	
Exists	$\exists x.\ \varphi$	

Conjunction $\varphi_1 \wedge \varphi_2$ as a Refinement Type

$$(\varphi_1, \varphi_2)$$

Conjunction $\varphi_1 \wedge \varphi_2$ as a Refinement Type

$$\frac{\Gamma \vdash \texttt{e :: } (\varphi_1, \varphi_2)}{\Gamma \vdash \texttt{case e of \{(x,y) -> x\} :: } \varphi_1}$$

(i.e., ∧-L-Elimination)

Conjunction $\varphi_1 \wedge \varphi_2$ as a Refinement Type

$$\frac{\Gamma \vdash \texttt{e :: } (\varphi_1, \varphi_2)}{\Gamma \vdash \texttt{case e of \{(x,y) -> y\} :: } \varphi_2}$$

(i.e., ∧-R-Elimination)

Conjunction $\varphi_1 \wedge \varphi_2$ as a Refinement Type

$$\frac{\Gamma \vdash e_1 :: \varphi_1 \qquad \Gamma \vdash e_2 :: \varphi_2}{\Gamma \vdash (e_1, e_2) :: (\varphi_1, \varphi_2)}$$

(i.e., $\wedge$-Introduction)

	Logical Formula	Refinement Type
Native Term	e	$\{v{:}() \mid e\}$
Conjunction	$\varphi_1 \wedge \varphi_2$	(φ_1, φ_2)
Disjunction	$\varphi_1 \vee \varphi_2$	
Implication	$\varphi_1 \Rightarrow \varphi_2$	
Forall	$\forall x. \varphi$	
Exists	$\exists x. \varphi$	

	Logical Formula	**Refinement Type**
Native Term	e	$\{v{:}() \mid e\}$
Conjunction	$\varphi_1 \land \varphi_2$	(φ_1, φ_2)
Disjunction	$\varphi_1 \lor \varphi_2$	**??**
Implication	$\varphi_1 \Rightarrow \varphi_2$	
Forall	$\forall\ x.\ \varphi$	
Exists	$\exists\ x.\ \varphi$	

Disjunction $\varphi_1 \vee \varphi_2$ as a Refinement Type

$$\text{Either } \varphi_1 \ \varphi_2$$

```
data Either a b = Left a | Right b
```

	Logical Formula	**Refinement Type**
Native Term	e	$\{v:() \mid e\}$
Conjunction	$\varphi_1 \wedge \varphi_2$	(φ_1, φ_2)
Disjunction	$\varphi_1 \vee \varphi_2$	Either φ_1 φ_2
Implication	$\varphi_1 \Rightarrow \varphi_2$	
Forall	$\forall$ x. φ	
Exists	$\exists$ x. φ	

	Logical Formula	**Refinement Type**
Native Term	e	$\{v:() \mid e\}$
Conjunction	$\varphi_1 \wedge \varphi_2$	(φ_1, φ_2)
Disjunction	$\varphi_1 \vee \varphi_2$	Either φ_1 φ_2
Implication	$\varphi_1 \Rightarrow \varphi_2$	**??**
Forall	$\forall x.\ \varphi$	
Exists	$\exists x.\ \varphi$	

Implication $\varphi_1 \Rightarrow \varphi_2$ as a Refinement Type

$$\varphi_1 \rightarrow \varphi_2$$

Implication $\varphi_1 \Rightarrow \varphi_2$ as a Refinement Type

$$\frac{\Gamma,\ \texttt{x}:\ \varphi_1 \vdash \texttt{e} :: \varphi_2}{\Gamma \vdash \texttt{\textbackslash x. e} :: \varphi_1 \rightarrow \varphi_2}$$

(i.e., $\Rightarrow$-Introduction)

Implication $\varphi_1 \Rightarrow \varphi_2$ as a Refinement Type

$$\frac{\Gamma \vdash e :: \varphi_1 \to \varphi_2 \qquad \Gamma \vdash e_x :: \varphi_1}{\Gamma \vdash e\ e_x :: \varphi_2}$$

(i.e., $\Rightarrow$-Elimination)

	Logical Formula	Refinement Type
Native Term	e	$\{v{:}() \mid e\}$
Conjunction	$\varphi_1 \wedge \varphi_2$	(φ_1, φ_2)
Disjunction	$\varphi_1 \vee \varphi_2$	Either φ_1 φ_2
Implication	$\varphi_1 \Rightarrow \varphi_2$	$\varphi_1 \rightarrow \varphi_2$
Forall	$\forall x.\ \varphi$	??
Exists	$\exists x.\ \varphi$	

$\forall$ x. φ as a Refinement Type

$$x{:}\varphi_x \rightarrow \varphi$$

$\forall$ x. φ as a Refinement Type

$$\frac{\Gamma,\ x : \varphi_x \vdash e :: \varphi}{\Gamma \vdash \backslash x.\ e :: x{:}\varphi_x \rightarrow \varphi}$$

(i.e., $\forall$-Introduction)

$\forall$ x. φ as a Refinement Type

$$\frac{\Gamma \vdash e :: x{:}\varphi_x \to \varphi \qquad \Gamma \vdash e_x :: \varphi_x}{\Gamma \vdash e\ e_x :: \varphi[e_x/x]}$$

(i.e., $\forall$-Elimination)

	Logical Formula	**Refinement Type**
Native Term	e	$\{v{:}() \mid e\}$
Conjunction	$\varphi_1 \land \varphi_2$	(φ_1, φ_2)
Disjunction	$\varphi_1 \lor \varphi_2$	Either φ_1 φ_2
Implication	$\varphi_1 \Rightarrow \varphi_2$	$\varphi_1 \rightarrow \varphi_2$
Forall	$\forall$ x. φ	$x{:}\varphi_x \rightarrow \varphi$
Exists	$\exists$ x. φ	??

$\exists$ x. φ as a Refinement Type

$$(\mathtt{x}{:}\varphi_{\mathtt{x}},\ \varphi)$$

$\exists$ x. φ as a Refinement Type

$$\frac{\Gamma \vdash e_x :: \varphi_x \qquad \Gamma, x : \varphi_x \vdash e :: \varphi}{\Gamma \vdash (e_x, e) :: (x{:}\varphi_x, \varphi)}$$

(i.e., $\exists$-Introduction)

$\exists$ x. φ as a Refinement Type

$$\frac{\begin{array}{c}\Gamma \vdash e :: (x{:}\varphi_x,\ \varphi) \\ \Gamma,\ x{:}\varphi_x\ ,\ x_\varphi{:}\varphi \vdash e_0 :: \varphi_0\end{array}}{\Gamma \vdash \texttt{case } e \texttt{ of } \{(x, x_\varphi) \texttt{ -> } e_0\} :: \varphi_0}$$

(i.e., $\exists$-Elimination)

	Logical Formula	Refinement Type
Native Terms	e	$\{v:() \mid e\}$
Conjunction	$\varphi_1 \wedge \varphi_2$	(φ_1, φ_2)
Disjunction	$\varphi_1 \vee \varphi_2$	Either φ_1 φ_2
Implication	$\varphi_1 \Rightarrow \varphi_2$	$\varphi_1 \to \varphi_2$
Forall	$\forall$ x. φ	$x{:}\varphi_x \to \varphi$
Exists	$\exists$ x. φ	$(x{:}\varphi_x, \varphi)$

	Logical Formula	**Refinement Type**
Native Terms	e	$\{v{:}() \mid e\}$
Conjunction	$\varphi_1 \wedge \varphi_2$	(φ_1, φ_2)
Disjunction	$\varphi_1 \vee \varphi_2$	Either φ_1 φ_2
Implication	$\varphi_1 \Rightarrow \varphi_2$	$\varphi_1 \to \varphi_2$
Forall	$\forall\, x.\ \varphi$	$x{:}\varphi_x \to \varphi$
Exists	$\exists\, x.\ \varphi$	$(x{:}\varphi_x, \varphi)$

Example:

$$(\exists\, x.\ \forall\, y.\ p\ x\ y) \Rightarrow (\forall\, y.\ \exists\, x.\ p\ x\ y)$$

Example:

$$\boxed{(\exists\, x.\ \forall\, y.\ p\ x\ y) \Rightarrow (\forall\, y.\ \exists\, x.\ p\ x\ y)}$$

$$\boxed{\rightarrow}$$

Example:

$$\boxed{(\exists\, x.\ \forall\, y.\ p\ x\ y)} \Rightarrow (\forall\, y.\ \exists\, x.\ p\ x\ y)$$

$$\boxed{(x{:}a, \qquad\qquad\qquad\qquad)} \to$$

Example:

$$(\exists\, x.\ \boxed{\forall\, y.\ p\ x\ y}) \Rightarrow (\forall\, y.\ \exists\, x.\ p\ x\ y)$$

$$(x{:}a,\ \boxed{y{:}a \rightarrow \qquad\qquad\qquad\qquad}) \rightarrow$$

Example:

$$(\exists\, x.\ \forall\, y.\ \boxed{p\ x\ y}) \Rightarrow (\forall\, y.\ \exists\, x.\ p\ x\ y)$$

$$(x{:}a,\ y{:}a \rightarrow \boxed{\{v{:}\ ()\ \mid\ p\ x\ y\}}) \rightarrow$$

Example:

$$(\exists\, x.\ \forall\, y.\ p\ x\ y) \Rightarrow (\boxed{\forall\, y.\ \exists\, x.\ p\ x\ y})$$

$$(x{:}a,\ y{:}a \rightarrow \{v{:}\ ()\ \mid\ p\ x\ y\})\rightarrow \boxed{y{:}a \rightarrow \quad\quad\quad\quad\quad\quad}$$

Example:

$$(\exists\, x.\ \forall\, y.\ p\ x\ y) \Rightarrow (\forall\, y.\ \boxed{\exists\, x.\ p\ x\ y})$$

$$(x{:}a,\ y{:}a \to \{v{:}\ ()\ \mid\ p\ x\ y\}) \to y{:}a \to \boxed{(x{:}a, \qquad\qquad\qquad}\,)$$

Example:

$$(\exists\, x.\ \forall\, y.\ p\ x\ y) \Rightarrow (\forall\, y.\ \exists\, x.\ \boxed{p\ x\ y})$$

$$(x{:}a,\ y{:}a \rightarrow \{v{:}\ ()\ \mid\ p\ x\ y\}\ ) \rightarrow y{:}a \rightarrow (x{:}a,\ \boxed{\{v{:}()\ \mid\ p\ x\ y\}})$$

Example:

$$(\exists\, x.\ \forall\, y.\ p\ x\ y) \Rightarrow (\forall\, y.\ \exists\, x.\ p\ x\ y)$$

$$(x{:}a,\ y{:}a \to \{v{:}\ ()\ |\ p\ x\ y\}) \to y{:}a \to (x{:}a,\ \{v{:}()\ |\ p\ x\ y\})$$

Proof?

Theorem Proving in Haskell

I. Refinement Types

II. Theorems as Types

III. Proofs as Programs

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DEMO

Theorem Proving in Haskell

I. Refinement Types

II. Theorems as Types

III. Proofs as Programs

Looking for a PostDoc @IMDEA, Madrid

Theorem Proving in Haskell

I. Refinement Types

II. Theorems as Types

III. Proofs as Programs

Thanks!