Semantics of Probabilistic Automata via Coalgebra

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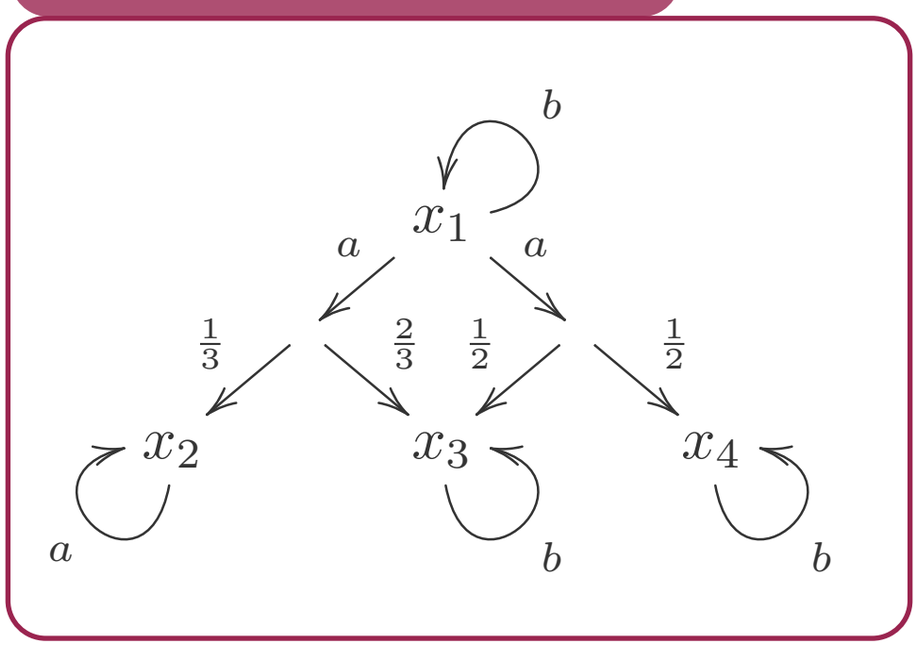
probabilistic automata

The different natures of PA

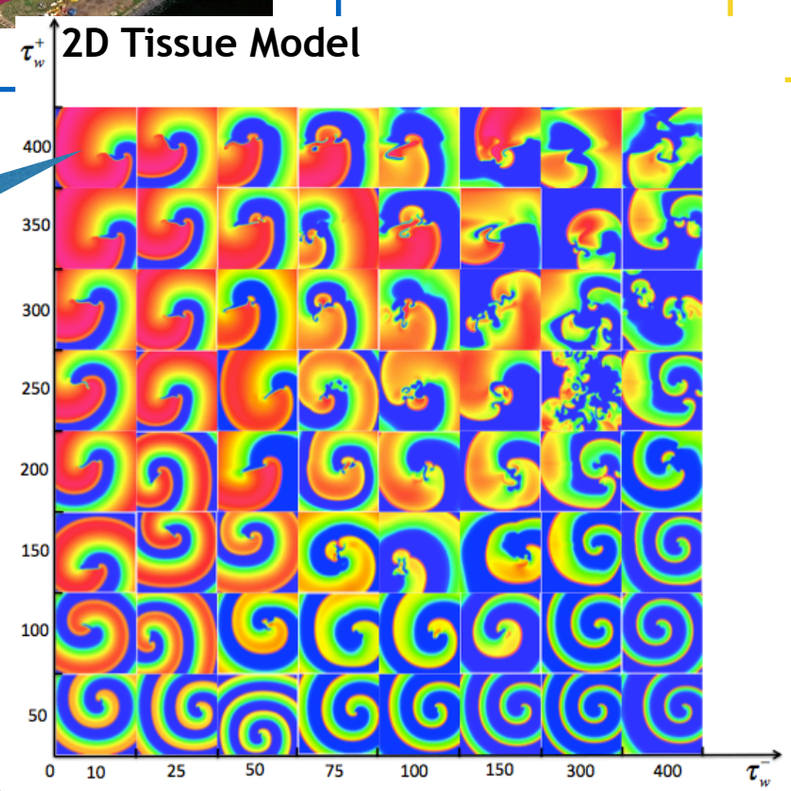
$X \xrightarrow{c} (\mathcal{P}\mathcal{D}X)^A$

we write $s \xrightarrow{a} \mu$ for $\mu \in c(s)(a)$

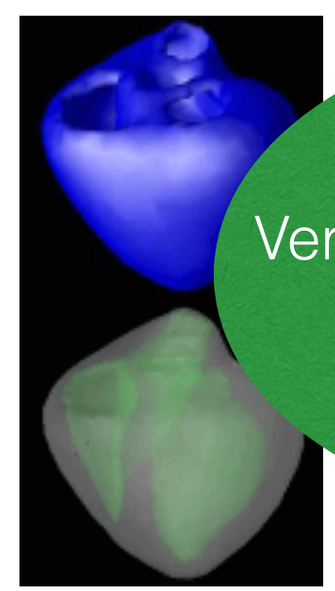


Where do they appear ?

2D Tissue Model



3D Organ Model



Model of a human heart Bartocci et al. TCS09, CAV11

Verification requires clear semantics

and suffers from state-space explosion

Probabilistic Automata

Can be given different semantics:

1. Bisimilarity
2. Convex bisimilarity
3. Distribution bisimilarity
4. Trace semantics

strong bisimilarity

probabilistic/ combined bisimilarity

belief-state bisimilarity

[Bonchi,Silva, S. CONCUR'17]

trace and testing theory

[Bonchi, S., Vignudelli LICS'19]

Behavioural Equivalences

LTS

$X \to (\mathcal{P}X)^A$

Two states are equivalent iff they admit the same traces (words).

trace equivalence

An equivalence relation $R \subseteq X \times X$ is a bisimulation of the LTS $X \to (\mathcal{P}X)^A$ iff whenever $(x, y) \in R$ for all $a \in A$

$$x \xrightarrow{a} x' \quad \Rightarrow \quad \exists y'.y \xrightarrow{a} y' \wedge (x', y') \in R.$$

Bisimilarity, denoted by $\sim$, is the largest bisimulation.

bisimilarity

Behavioural Equivalences

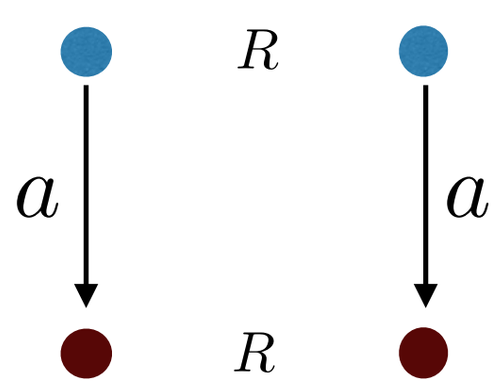
NFA

$X \to (\mathcal{P}X)^A$

Two states are equivalent iff they admit the same traces (words).

trace equivalence

bisimulation



largest bisimulation

bisimilarity

Probabilistic Automata

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[Bonchi,Silva, S. CONCUR'17]

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Bisimilarity

An equivalence relation R on the PA $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$ is a bisimulation iff whenever $(s,t) \in R$ for all $a \in A$ and $\mu \in \mathcal{D}X$

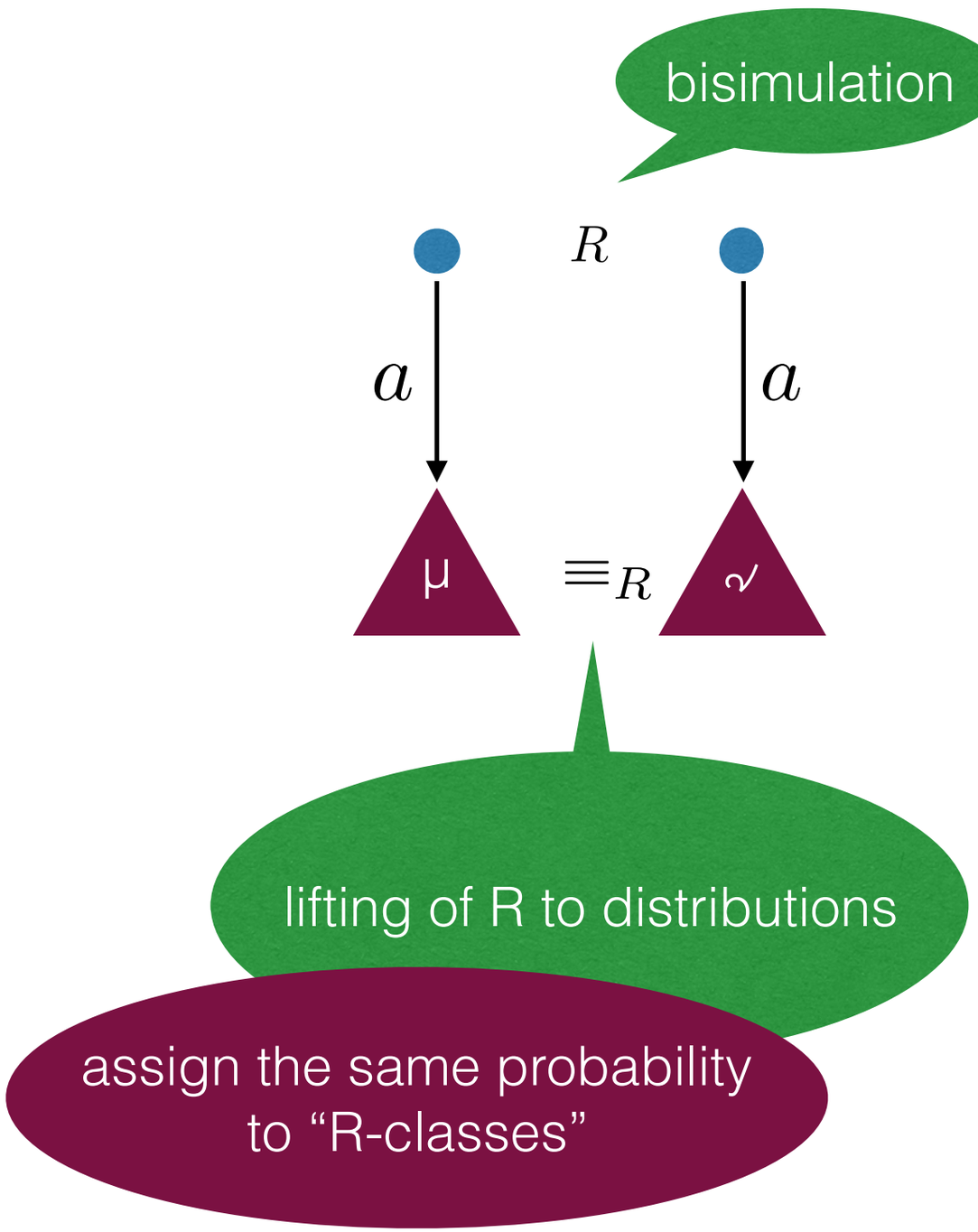
$$s \xrightarrow{a} \mu \Longrightarrow \exists \nu \in \mathcal{D}X.\ t \xrightarrow{a} \nu \wedge \mu \equiv_R \nu$$

where $\mu \equiv_R \nu$ iff $\mu[C] = \nu[C]$ for all R-equivalence classes C, with $\mu[C] = \sum_{x \in C} \mu(x)$.

Bisimilarity on $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$, denoted by $\sim$, is the largest bisimulation.

Bisimilarity

$\sim$ largest bisimulation



Convex bisimilarity

An equivalence relation $R \subseteq X \times X$ is a convex bisimulation of the PA $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$ iff whenever $(x, y) \in R$, for all $a \in A$ and $\mu \in \mathcal{D}X$

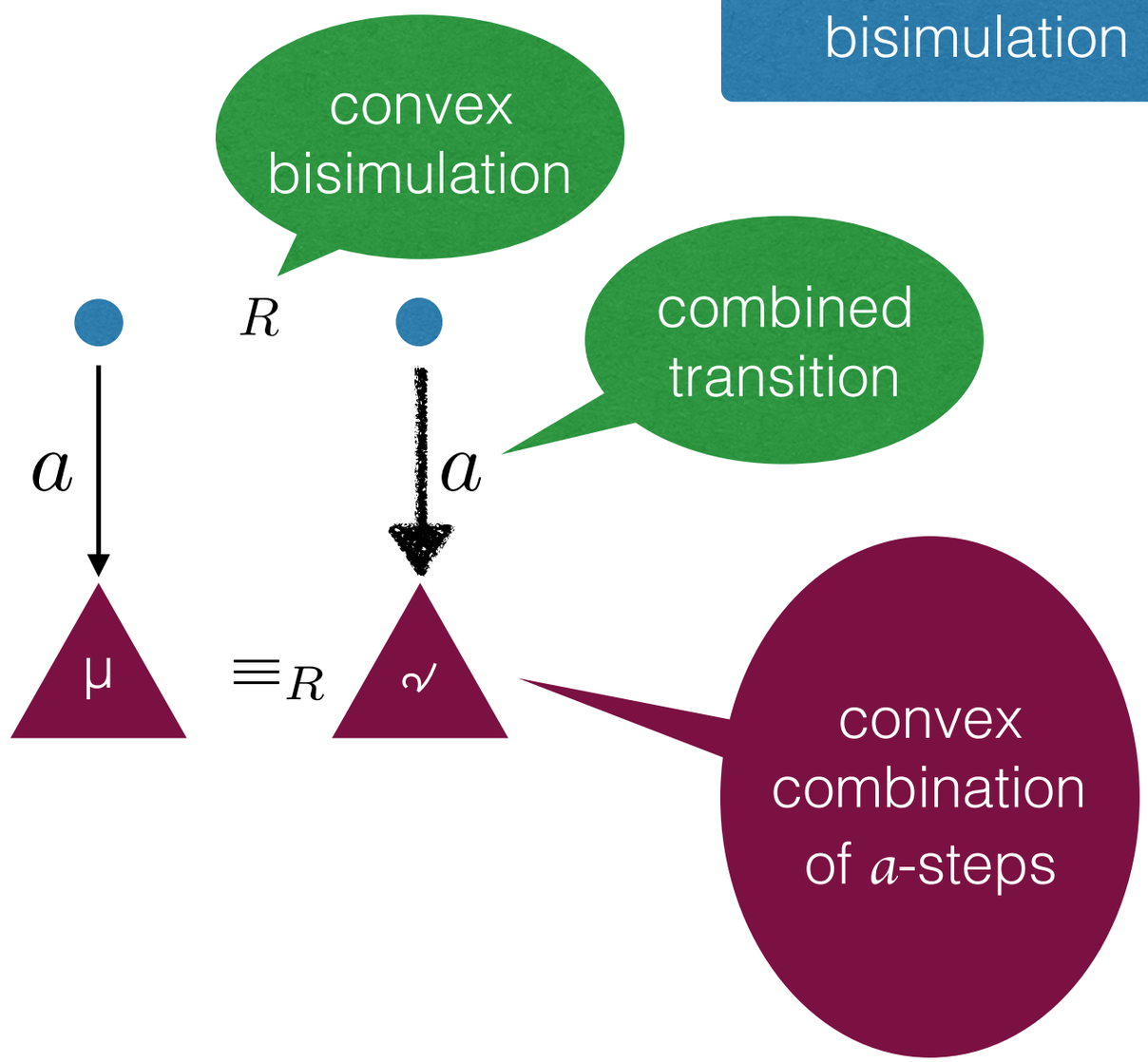
convex combination

$$x \xrightarrow{a} \mu \quad \Rightarrow \quad \exists \nu.\, \mu \equiv_R \nu \wedge \nu = \sum_{i=1}^{n} p_i \nu_i \wedge y \xrightarrow{a} \nu_i.$$

Convex bisimilarity on $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$, denoted by $\sim_c$, is the largest bisimulation.

Convex bisimilarity

$\sim_c$
largest convex bisimulation



Distribution bisimilarity

An equivalence relation R on the carrier of the belief-state transformer $c\colon \mathcal{D}X \to (\mathcal{P}\mathcal{D}X)^A$ is a distribution bisimulation iff whenever $(\mu, \nu) \in R$ for all $a \in A$

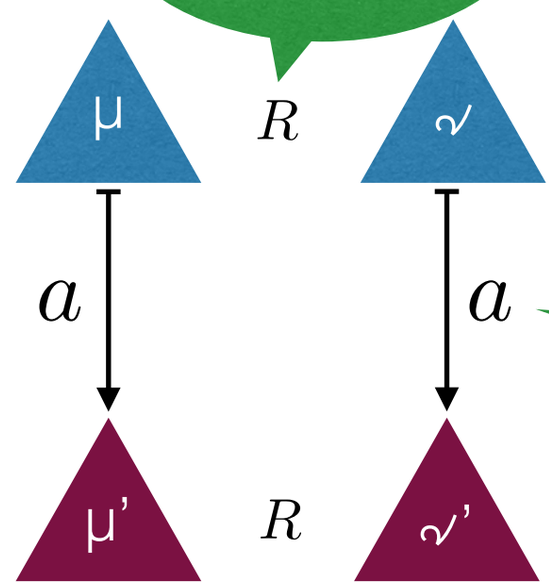
$$\mu \xrightarrow{a} \mu' \Longrightarrow \exists \nu' \in \mathcal{D}X.\ \nu \xrightarrow{a} \nu' \wedge (\mu', \nu') \in R.$$

Distribution bisimilarity on $c\colon \mathcal{D}X \to (\mathcal{P}\mathcal{D}X)^A$, denoted by $\sim_d$, is the largest distribution bisimulation.

Distribution bisimilarity

$\sim_d$
largest distribution bisimulation

distribution bisimulation



transition in the belief-state transformer

$\sim_d$
is LTS bisimilarity on the belief-state transformer

Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$$X \xrightarrow{c} FX$$

generic notion of behavioural equivalence

$\approx$

states

behaviour type

object in the base category $\mathbf{C}$

functor on the base category $\mathbf{C}$

form a category too

$\mathbf{CoAlg_C}(F)$

The category of F-coalgebras

$\mathbf{CoAlg_C}(F)$

behaviour-preserving maps

Objects = coalgebras

Arrows = coalgebra homomorphisms

$$X \xrightarrow{c} FX$$

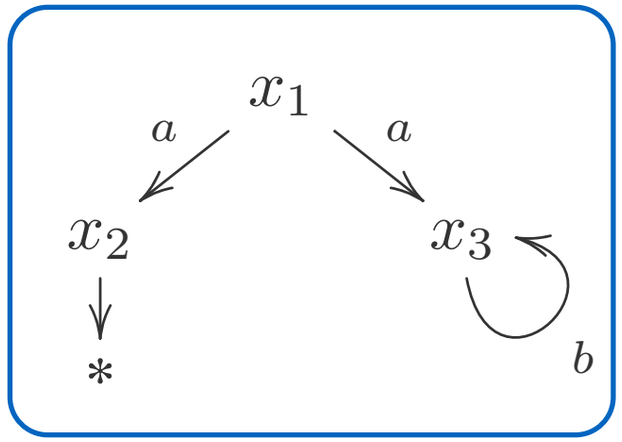
$$h\colon X \to Y \qquad \begin{tikzcd} X \arrow[r, "h"] \arrow[d, "c_X"'] & Y \arrow[d, "c_Y"] \\ FX \arrow[r, "Fh"'] & FY \end{tikzcd}$$

Two states $x, y \in X$ are behaviourally equivalent, notation $x \approx y$ iff there exists a coalgebra homomorphism $h\colon X \to Y$ from $c\colon X \to FX$ to some coalgebra $d\colon Y \to FY$ such that $h(x) = h(y)$.

Examples

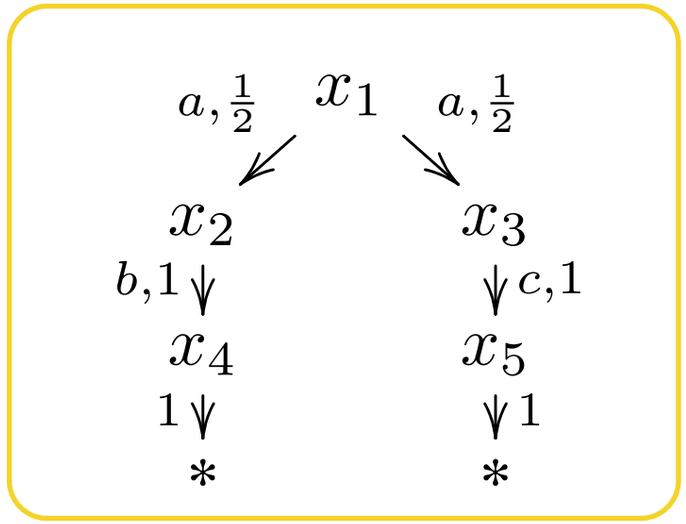
NFA

$X \to 2 \times (\mathcal{P}X)^A$



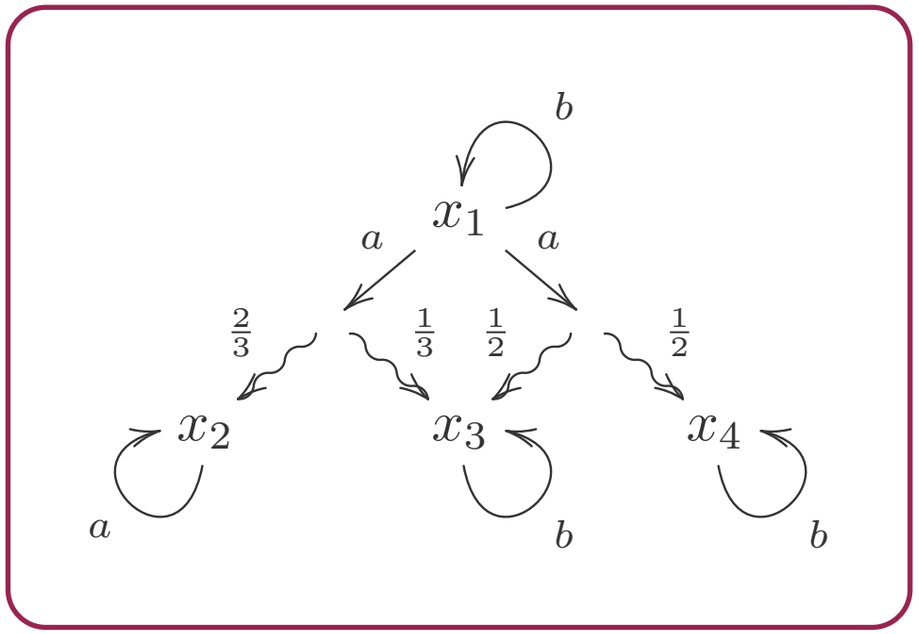
Generative PTS

$X \to \mathcal{D}(1 + A \times X)$



PA

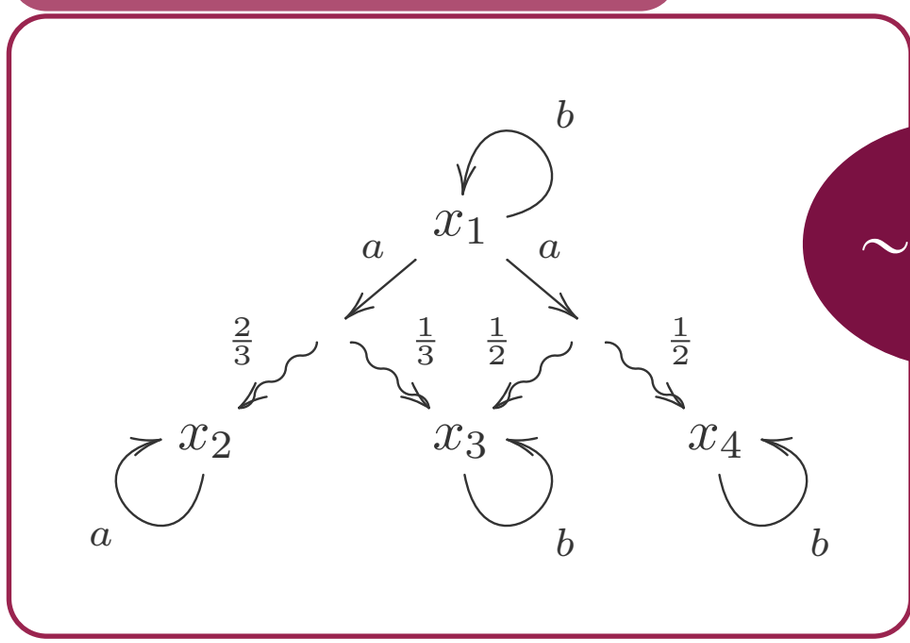
$X \to (\mathcal{P}\mathcal{D}X)^A$



all on
Sets

PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$



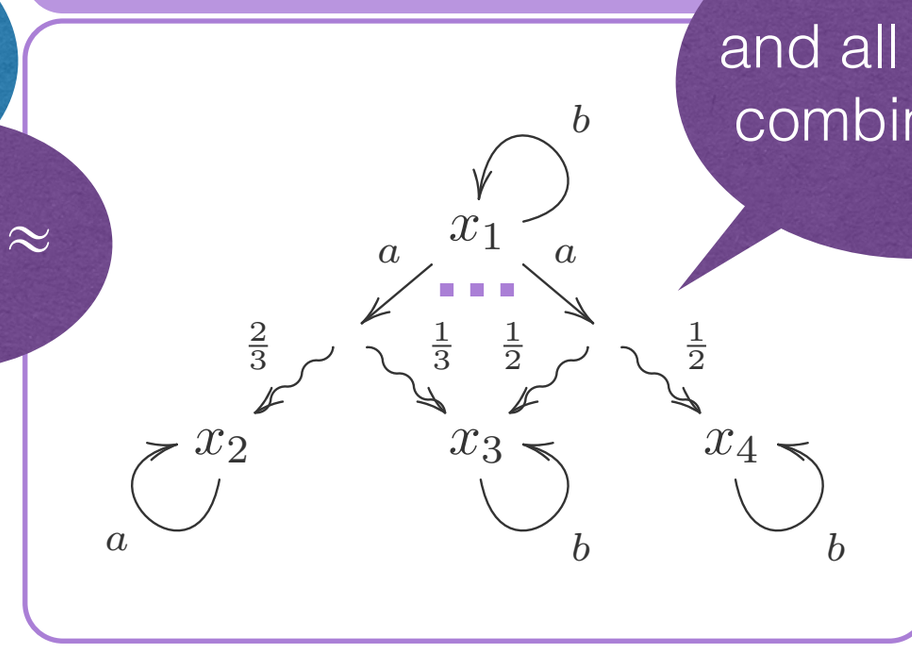
on **Sets**

$\sim \; = \; \approx$

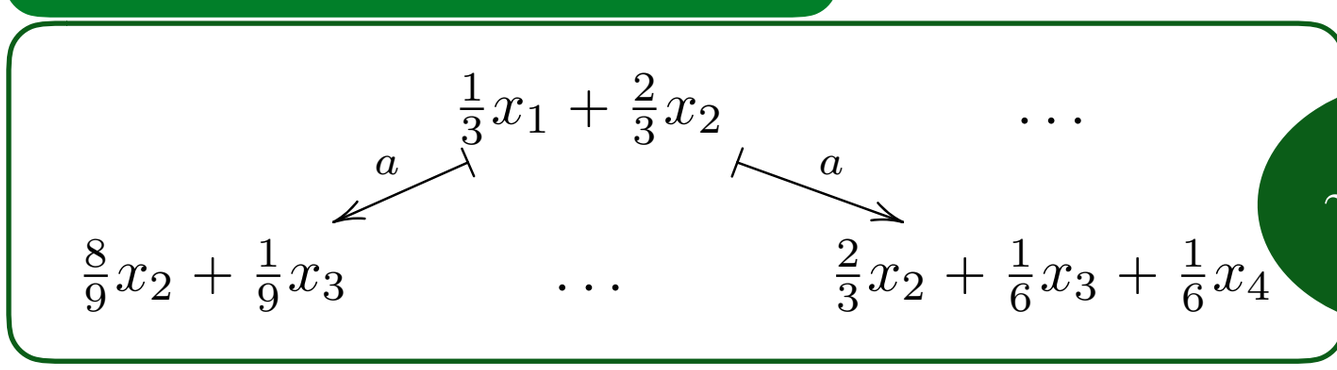
$\sim_c \; = \; \approx$

$X \to (\mathcal{C}X)^A$

and all convex combinations



$X \to (\mathcal{P}_c X+1)^A$



on convex algebras

$\mathcal{EM}(\mathcal{D})$

$\sim_d \; = \; \approx$

$\mathcal{EM}(\mathcal{D})$

Convex algebras

infinitely many finitary operations

convex combinations

- algebras $\quad (A, \sum_{i=1}^{n} p_i(-)_i)$ $\qquad p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$

- convex (affine) maps $\quad h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)$

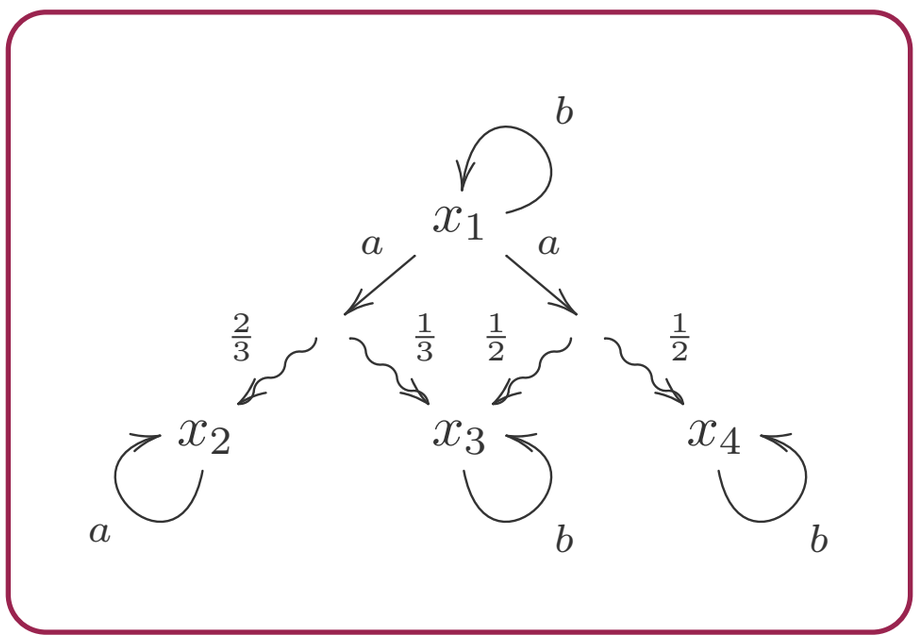
satisfying

- Projection $\quad \sum_{i=1}^{n} p_i a_i = a_k, \quad p_k = 1$
- Barycenter $\quad \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} p_{i,j} a_j\right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i p_{i,j}\right) a_j$

Belief-state transformer

PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



belief-state transformer

belief state

$$\frac{1}{3}x_1 + \frac{2}{3}x_2 \quad \ldots$$

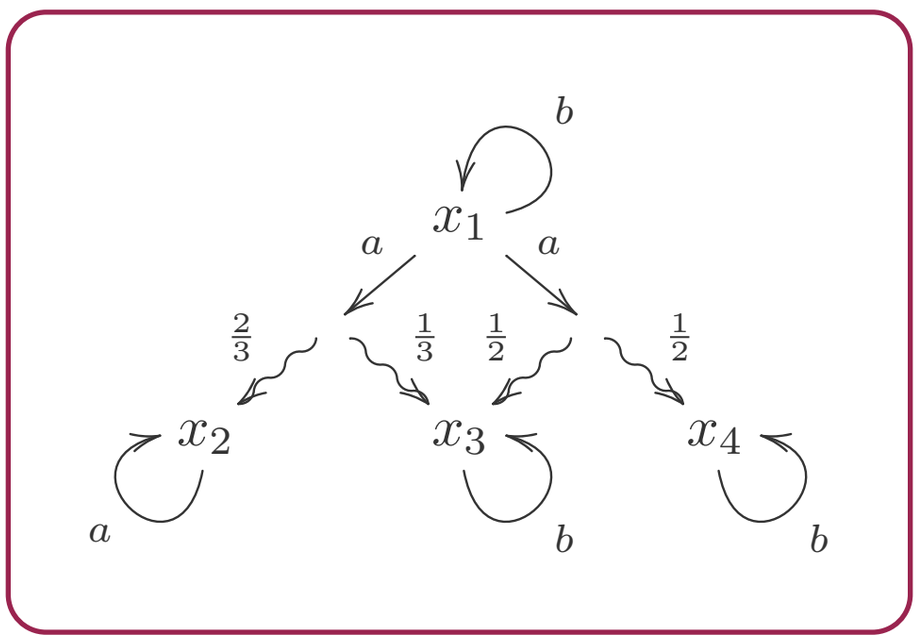
$$\xleftarrow{a} \qquad \xrightarrow{a}$$

$$\frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \quad \ldots \quad \frac{8}{9}x_2 + \frac{1}{9}x_3$$

Belief-state transformer

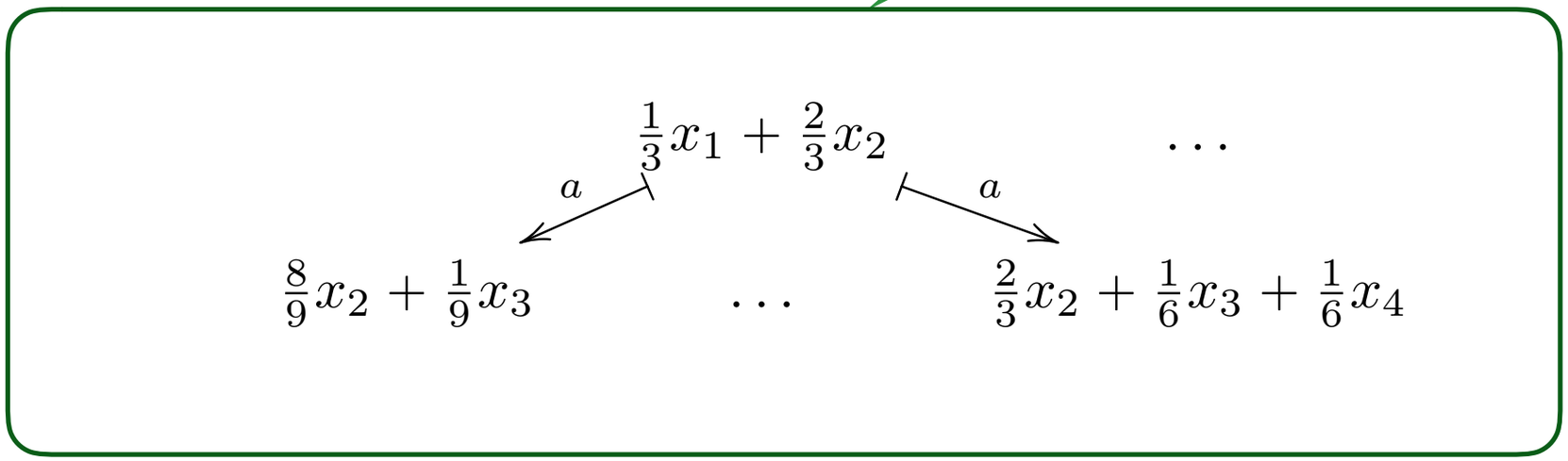
PA

$X \to (\mathcal{P}\mathcal{D}X)^A$

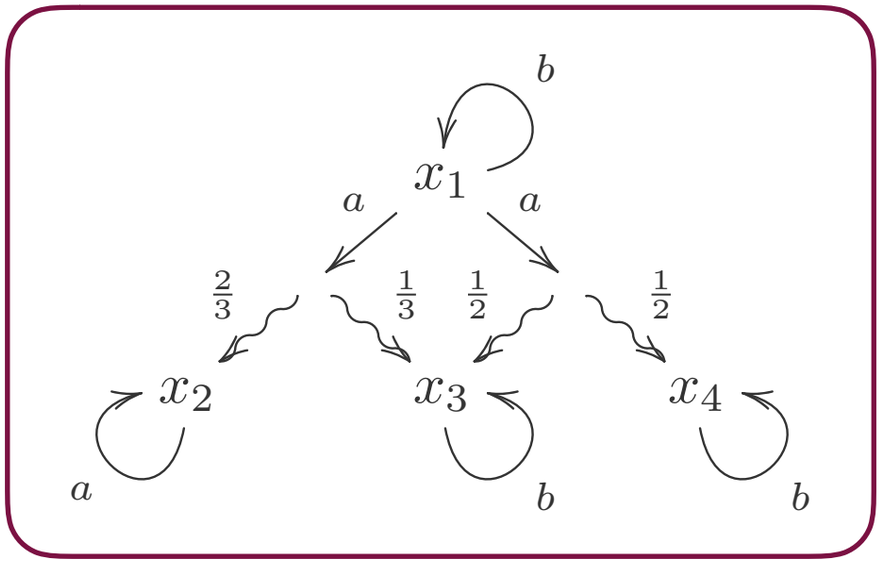


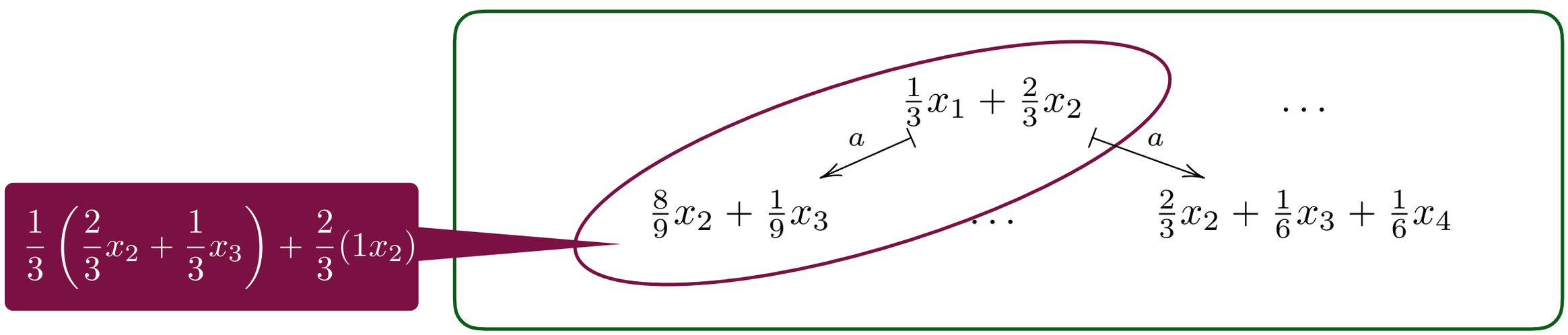
how does it emerge?

what is it?

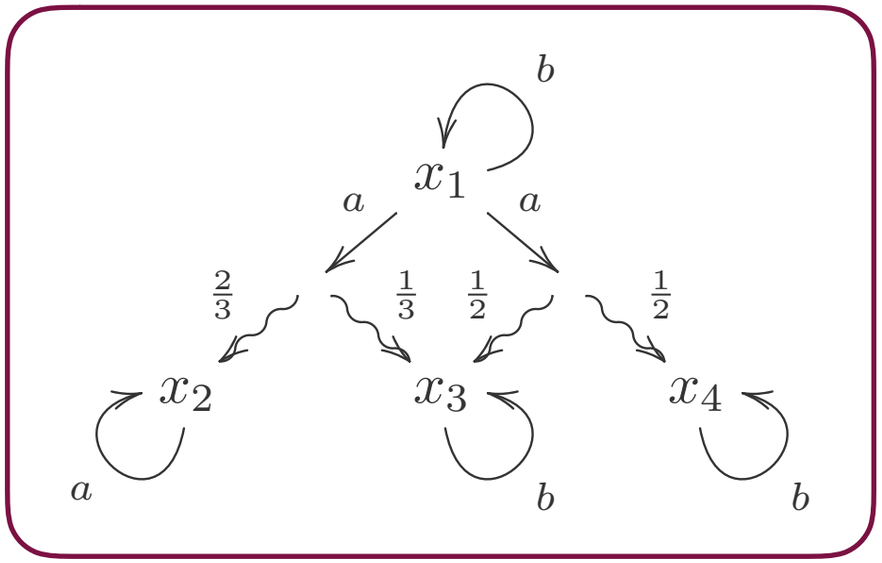


Belief-state transformer





Belief-state transformer



very infinite
LTS on belief states

$$\frac{1}{3}x_1 + \frac{2}{3}x_2 \qquad \dots$$

a $\qquad$ a

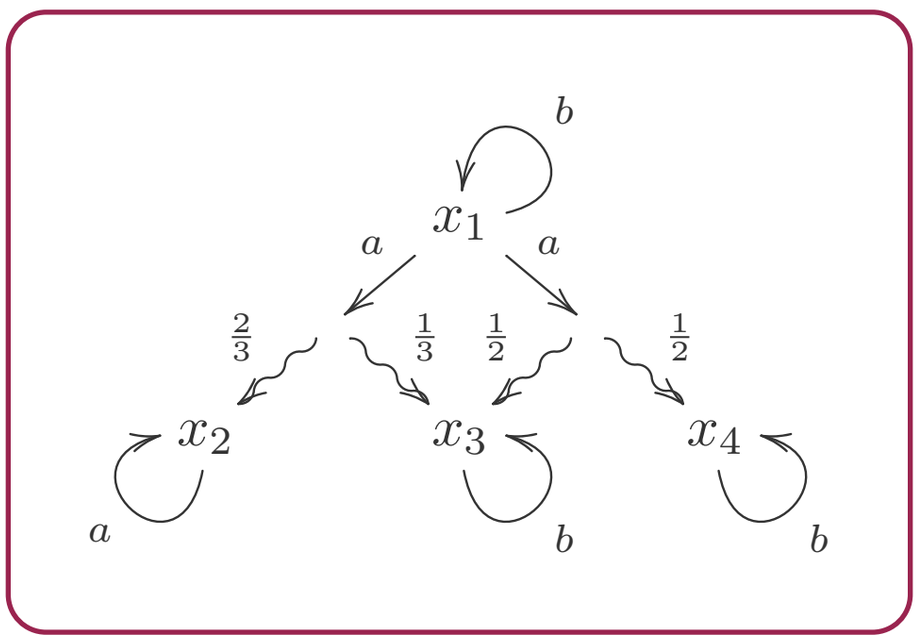
$$\frac{8}{9}x_2 + \frac{1}{9}x_3 \qquad \dots \qquad \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4$$

$$\frac{1}{3}\left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

Belief-state transformer

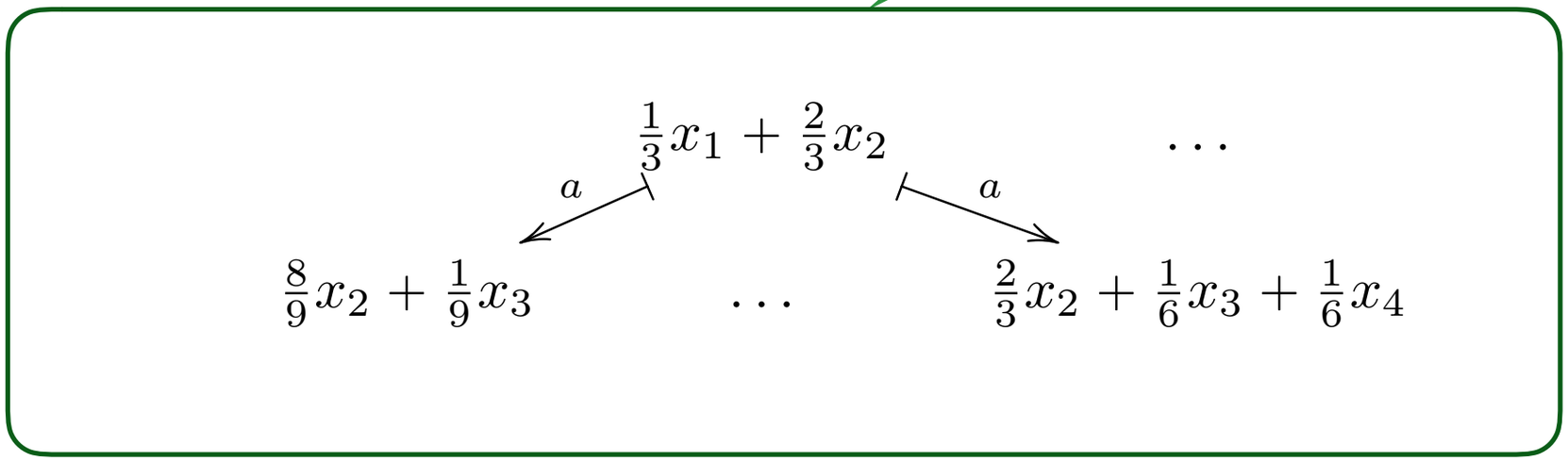
PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



how does it emerge?

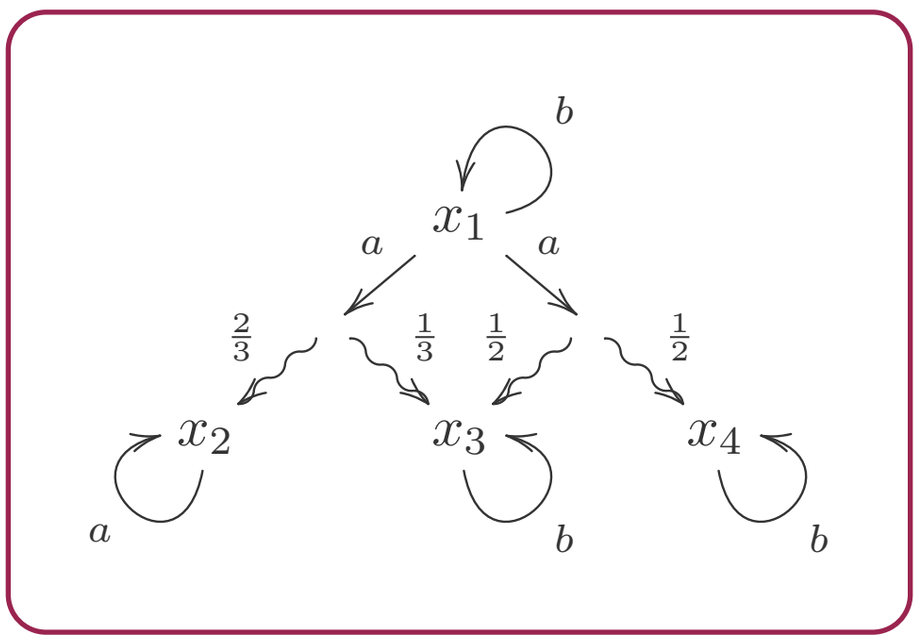
what is it?



Belief-state transformer

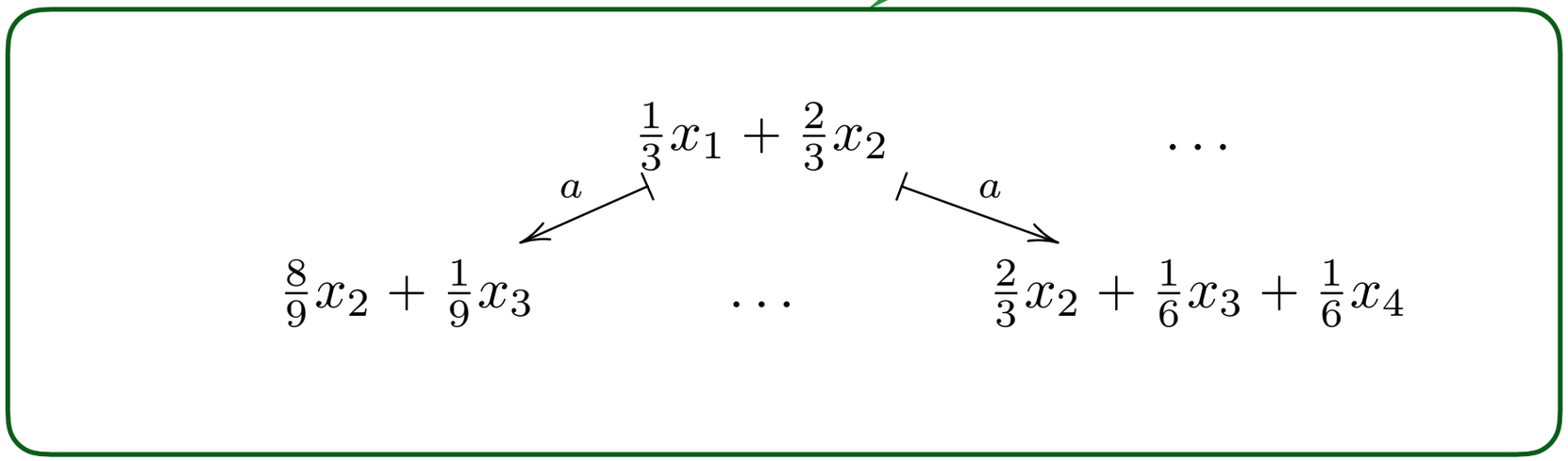
PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



how does it emerge?

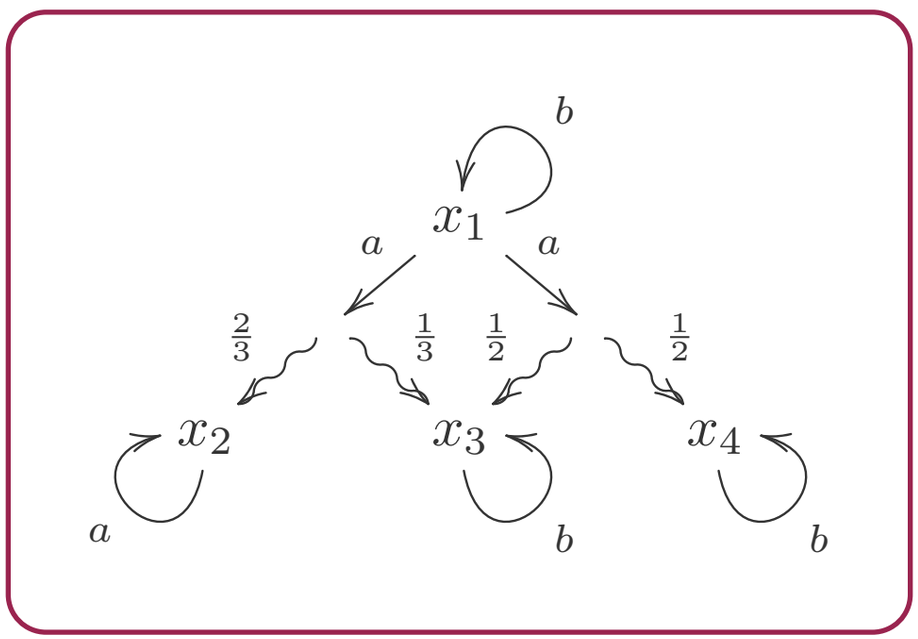
coalgebra over free convex algebra



Belief-state transformer

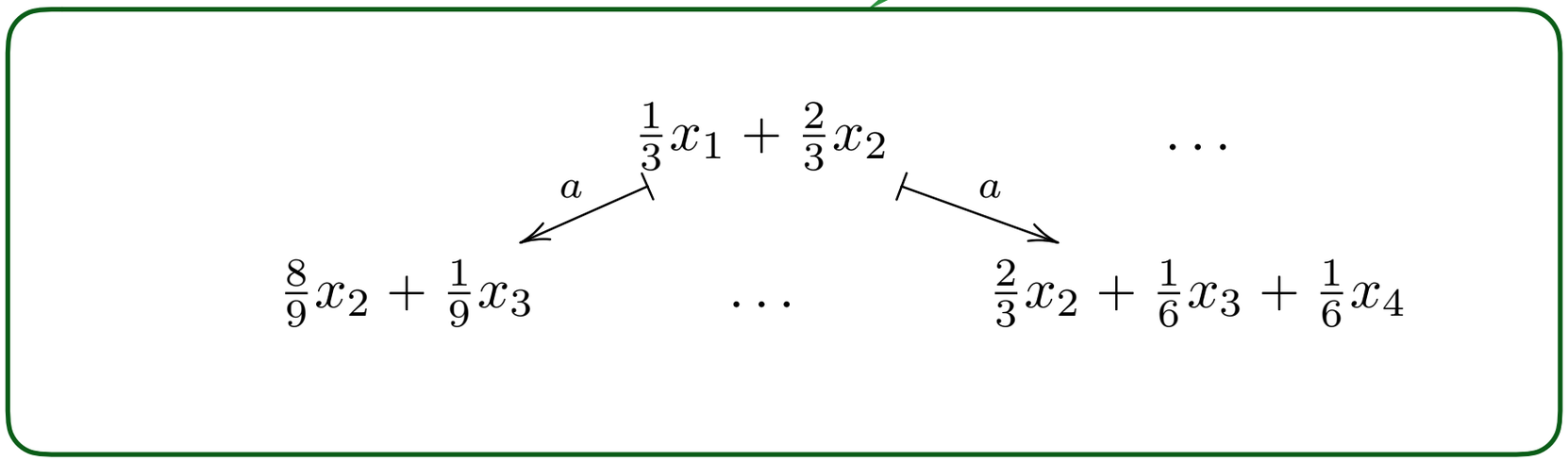
PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



via a generalised determinisation

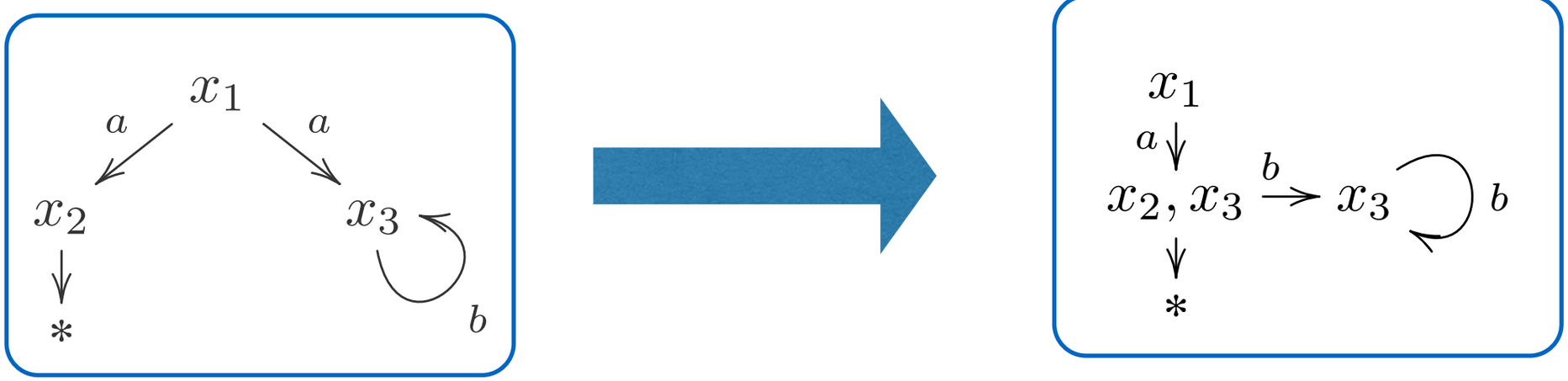
coalgebra over free convex algebra



Determinisations

NFA

$X \to 2 \times (\mathcal{P}X)^A$

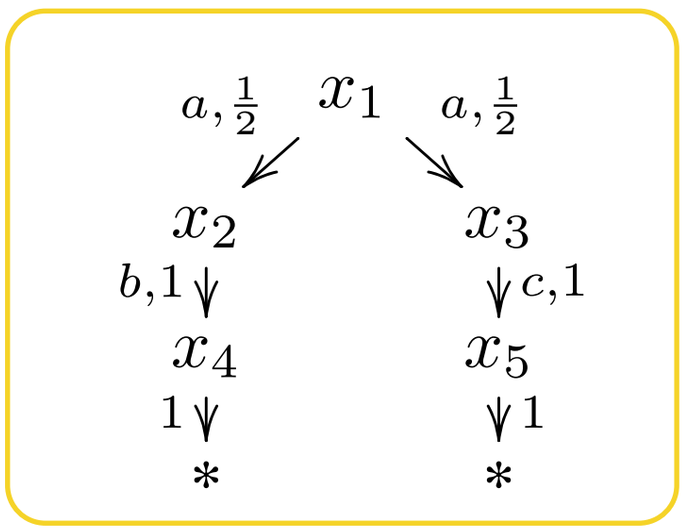


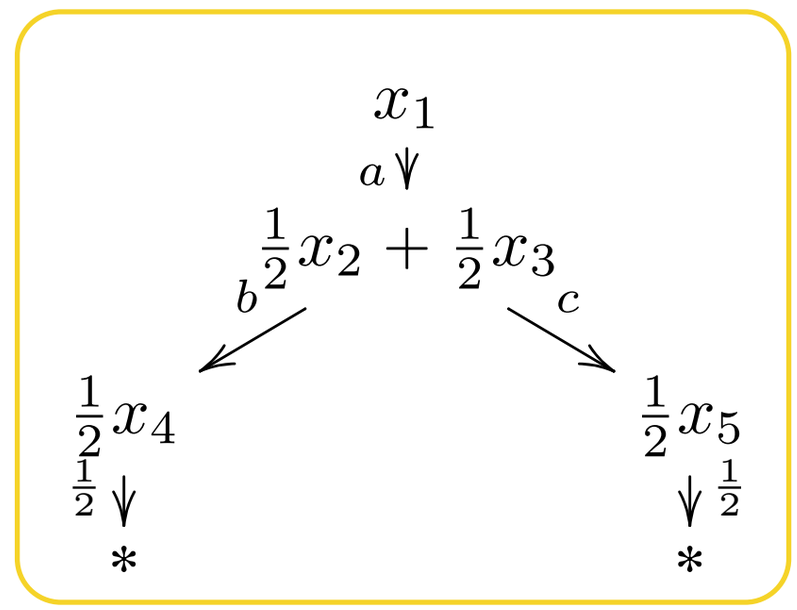
[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations

Generative PTS

$X \to \mathcal{D}(1 + A \times X)$





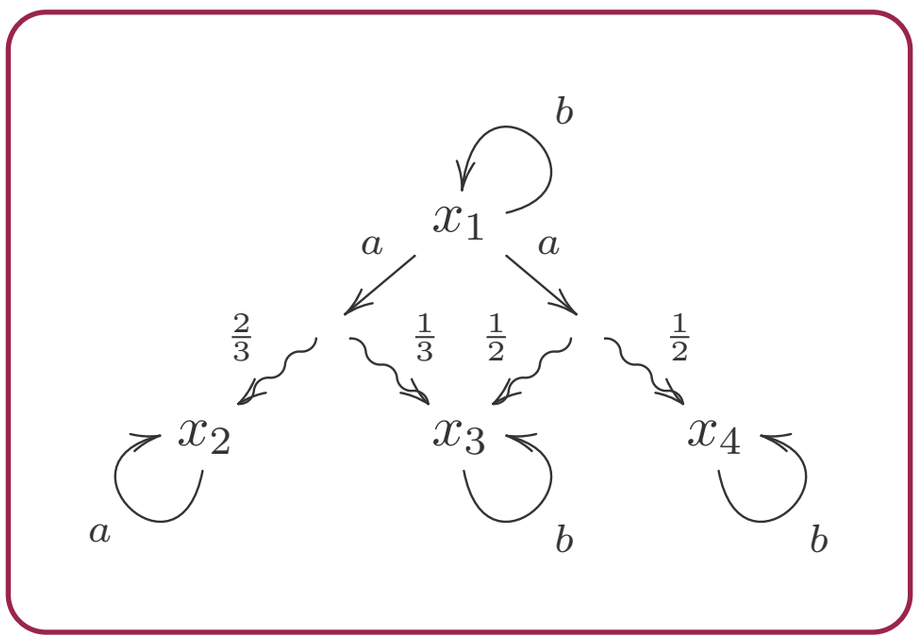
[Silva, S. MFPS'11]

[Jacobs, Silva, S. JCSS'15]

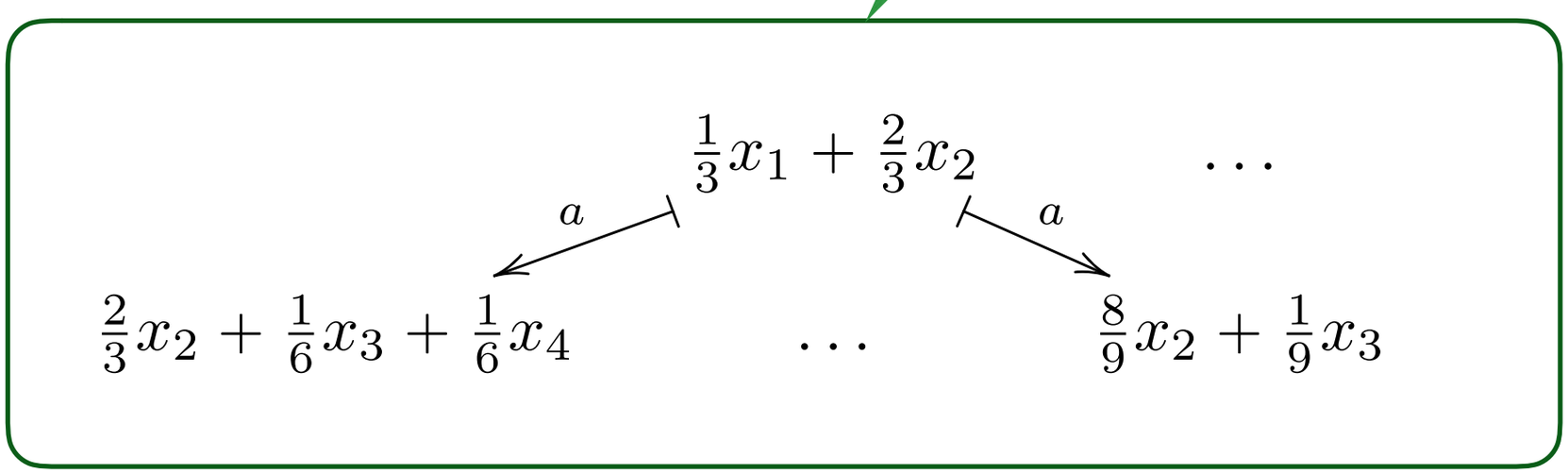
Determinisations

PA

$X \to (\mathcal{P}\mathcal{D}X)^A$

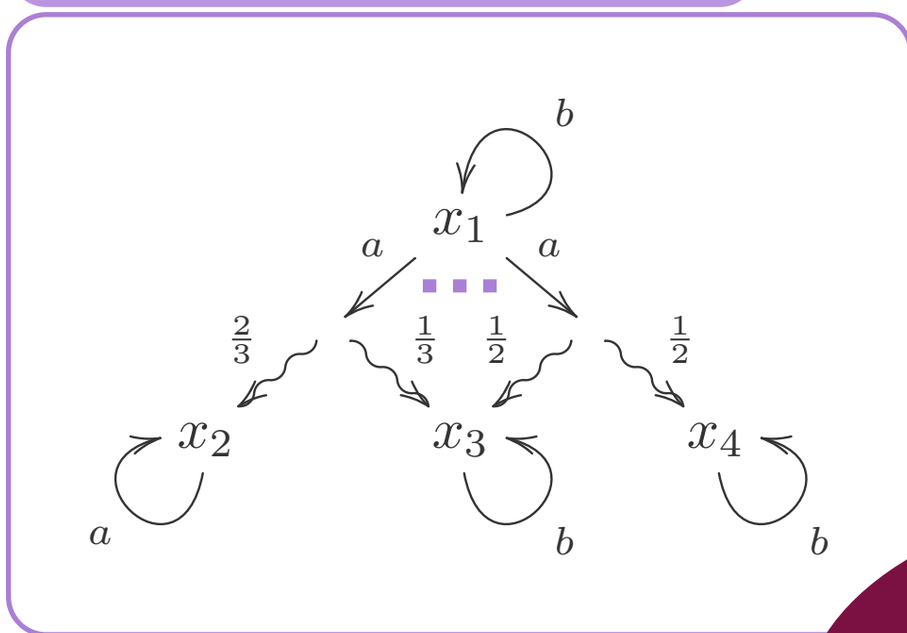


belief-state transformer



Determinisations

$X \to (\mathcal{C}X)^A$



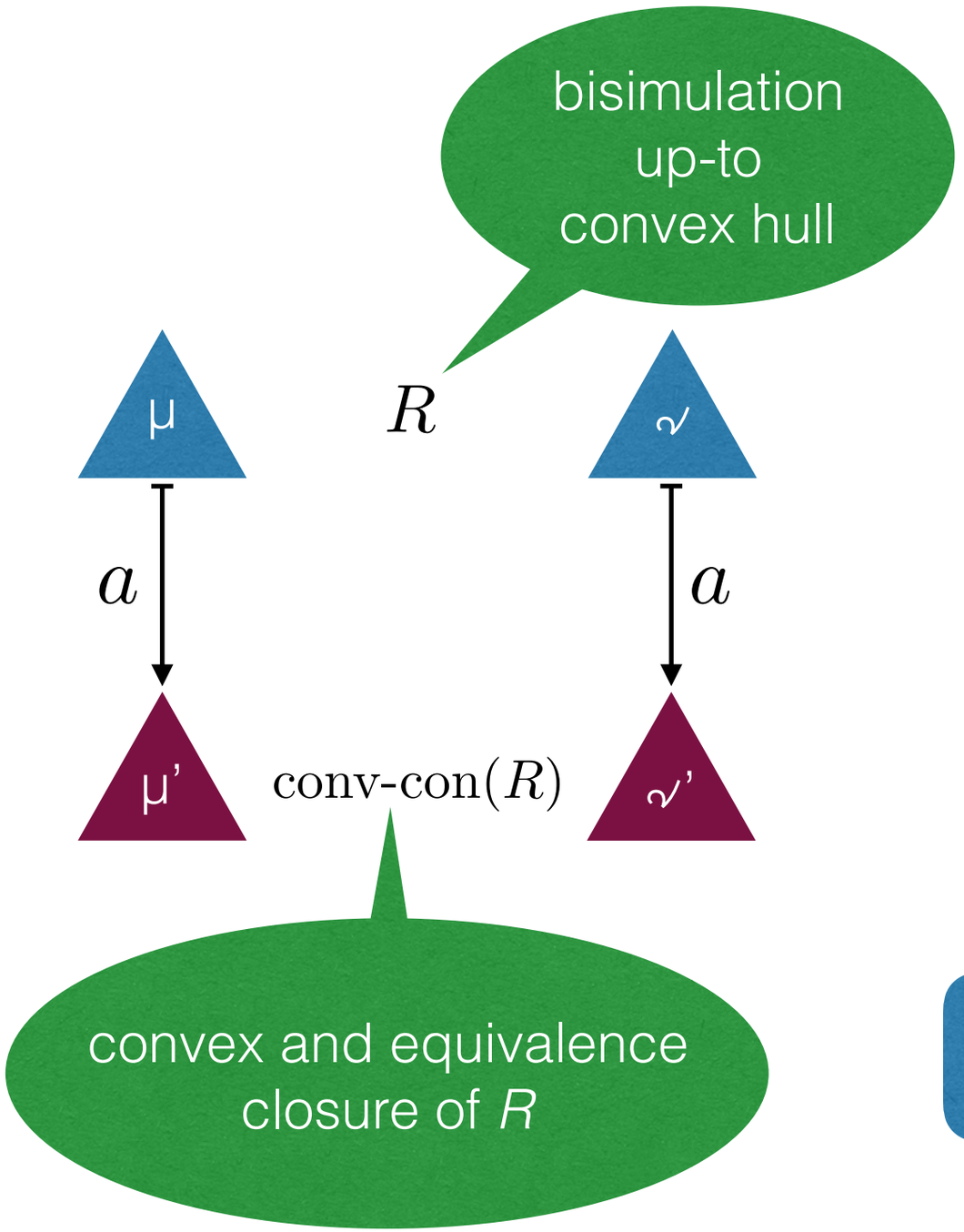
$$x_1 \xrightarrow{a} \left(\tfrac{2}{3}x_2 + \tfrac{1}{3}x_3\right) \oplus \left(\tfrac{1}{2}x_3 + \tfrac{1}{2}x_4\right)$$

Theory of traces for PA @LICS

LTS on a convex semilattice

Thank You !

Coinductive proof method for distribution bisimilarity



to prove $\mu \sim_d \nu$ it suffices to find a bisimulation up-to convex hull R with $\mu \; R \; \nu$

there always exists a finite one!

[S., Woracek JPAA'15]