Well-quasi-orders in Logic

Sylvain Schmitz

Panhellenic Logic Symposium, June 29, 2019

Outline

well-quasi-orders (wqo):

- robust notion
- selection of applications:
  verification algorithm termination

  proof theory relevance logic

  finite model theory preservation theorems

  database theory certain answers

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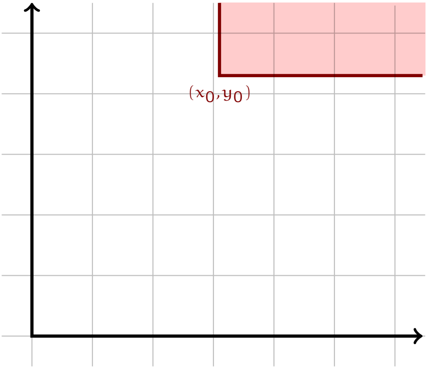
A One-Player Game

- over $\mathbb{Q}_{\geqslant 0} \times \mathbb{Q}_{\geqslant 0}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



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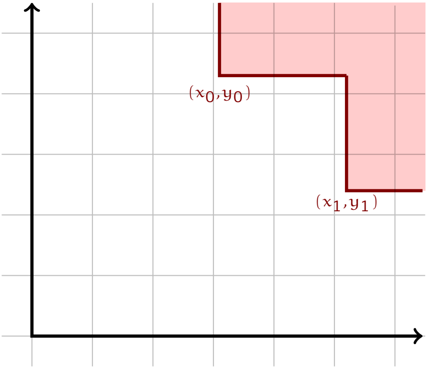

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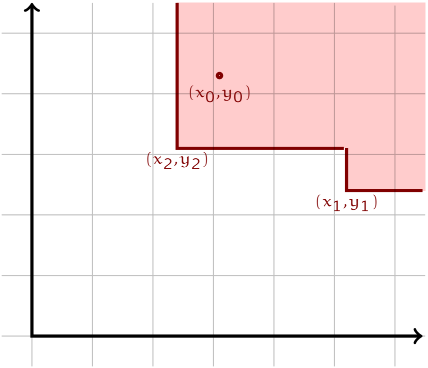
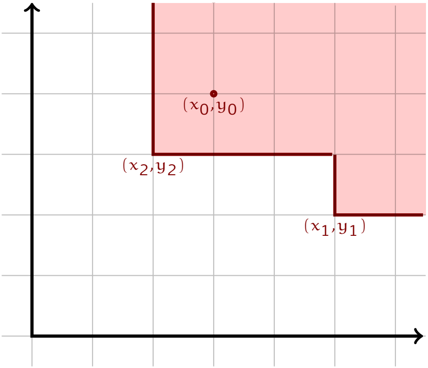


- Can Eloise win, i.e. play indefinitely?

If $(x_0, y_0) \neq (0,0)$, then choosing $(x_j, y_j) = (\frac{x_0}{2^j}, \frac{y_0}{2^j})$ wins.

A One-Player Game

- over $\mathbb{N} \times \mathbb{N}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



- Can Eloise win, i.e. play indefinitely?

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purple if $x_i > x_j$ but $y_i \leqslant y_j$,

red if $x_i > x_j$ and $y_i > y_j$,

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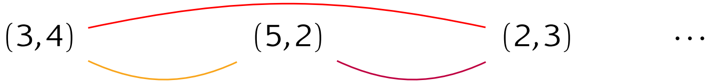

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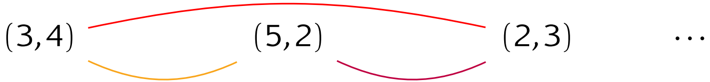

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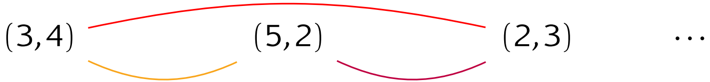

By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices. In all cases, it implies the existence of an infinite decreasing sequence in $\mathbb{N}$, a contradiction.

Well-Quasi-Orders

- multiple equivalent definitions
- algebraic constructions

Well-Quasi-Orders

- multiple equivalent definitions: $(X, \leqslant)$ wqo iff
  - bad sequences are finite: $x_0, x_1, \ldots$ is bad if $\forall i < j, x_i \not\leqslant x_j$
  - $\leqslant$ is well-founded and has no infinite antichains
  - finite basis property: $\emptyset \subsetneq U \subseteq X$ has at least one and finitely many minimal elements
  - ascending chain condition: any chain $U_0 \subsetneq U_1 \subsetneq \cdots$ of upwards-closed sets is finite
  - etc.
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Well-Quasi-Orders

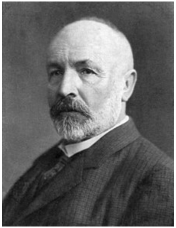
- multiple equivalent definitions
- algebraic constructions
  - Cartesian products (Dickson's Lemma),
  - finite sequences (Higman's Lemma),
  - disjoint sums,
  - finite sets with Hoare's quasi-ordering,
  - finite trees (Kruskal's Tree Theorem),
  - graphs with minors (Robertson and Seymour's Graph Minor Theorem),
  - etc.

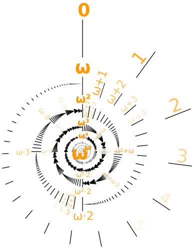
Example: Ordinals

ordinal: well-founded linear order

bad sequences are descending sequences:

$$\alpha \not\leq \beta \text{ iff } \alpha > \beta$$





Example: Dickson's Lemma

Lemma (Dickson 1913)

If $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ are two wqos, then $(X \times Y, \leqslant_\times)$ is a wqo, where $\leqslant_\times$ is the product ordering:

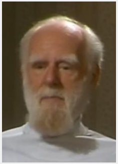
$$\langle x, y\rangle \leqslant_\times \langle x', y'\rangle \overset{\text{def}}{\Leftrightarrow} x \leqslant_X x' \wedge y \leqslant_Y y' \,.$$

Example

- $(\mathbb{N}^d, \leqslant_\times)$ using the product ordering
- $(\mathbb{M}(X), \leqslant_m)$ for finite multiset embedding over finite X

Example: Higman's Lemma

Lemma (Higman 1952)



If $(X, \leqslant)$ *is a wqo, then* $(X^*, \leqslant_*)$ *is a wqo where* $\leqslant_*$ *is the subword embedding ordering:*

$$a_1 \cdots a_m \leqslant_* b_1 \cdots b_n \overset{\text{def}}{\Leftrightarrow} \begin{cases} \exists 1 \leqslant i_1 < \cdots < i_m \leqslant n, \\ \bigwedge_{j=1}^{m} a_j \leqslant_A b_{i_j} \ . \end{cases}$$

Example

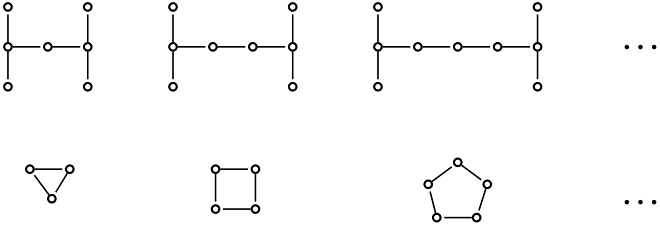
$$aba \leqslant_* baaacabbab$$

Example: Bounded Tree-Depth

Lemma (Ding 1992)

For all k, $(\text{Graphs} \setminus \uparrow P_k, \subseteq)$ *is wqo.*

Non-Examples



Application: Algorithm Termination

```
SIMPLE (a,b)
c ⟵ 1
while a > 0 ∧ b > 0
    ⟨a,b,c⟩ ⟵ ⟨a−1,b,2c⟩
  or
    ⟨a,b,c⟩ ⟵ ⟨2c,b−1,1⟩
end
```

$\langle a_0,b_0,c_0\rangle$
$\langle a_1,b_1,c_1\rangle$
$\vdots$
$\langle a_i,b_i,c_i\rangle$
$\vdots$
$\langle a_j,b_j,c_j\rangle$

- in any execution, $\langle a_0,b_0\rangle,\ldots,\langle a_n,b_n\rangle$ is a bad sequence over $(\mathbb{N}^2,\leqslant_\times)$,
- $(\mathbb{N}^2,\leqslant_\times)$ is a wqo: all the runs are finite

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Quasi-ranking Function

Definition

A function f from Confs to a wqo $(X, \leqslant)$ is a quasi-ranking function if, for all executions $c_0, c_1, \ldots, \forall i < j, f(c_i) \not\leqslant f(c_j)$.

Proposition

If an algorithm has a quasi-ranking function, then it terminates.

Proof.

The sequence of ranks $f(c_0), f(c_1), \ldots$ is a bad sequence over the wqo $(X, \leqslant)$. □

c.f. Podelski & Rybalchenko's transition invariants

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Application: Relevance Logic

Example $(A \to (B \to A))$

"if it's raining (A), then if your favorite color is green (B) then it's raining (A)"

A theorem in classical logic, not in relevance logic.

Gentzen-style sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

$$\frac{}{A \vdash A}\ (\mathrm{Id}) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ (\mathrm{C})$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\ (\to_L) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to_R)$$

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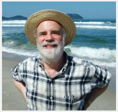
Problem (provability)

Given a sequent $\Gamma \vdash A$, is it provable?

Theorem (Kripke 1959)

Provability is decidable in implicational relevance logic.



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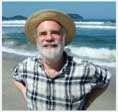

Proof Sketch.

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- subformula property
- irredundant proof searches
  - (C) and $(\to_R)$ commute: (C)'s only below a $(\to_L)$
  - rewrite proofs to apply (C) whenever possible
- irredundant proof branches are bad sequences for contraction
- ...which is wqo over the subformulæ of $\Gamma \vdash A$ □

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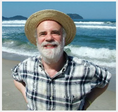

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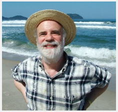

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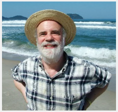

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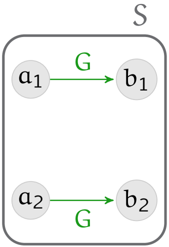
Application: Preservation Theorems

Formula $\varphi(x,y) = \exists z. x \xrightarrow{G} y \wedge \neg(y \xrightarrow{R} z)$



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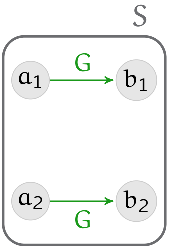

$\models \varphi(a_1, b_1)$

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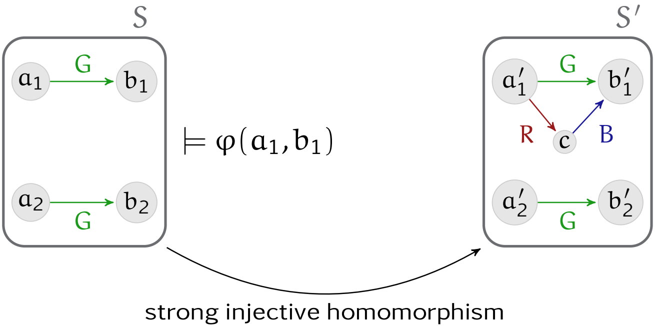

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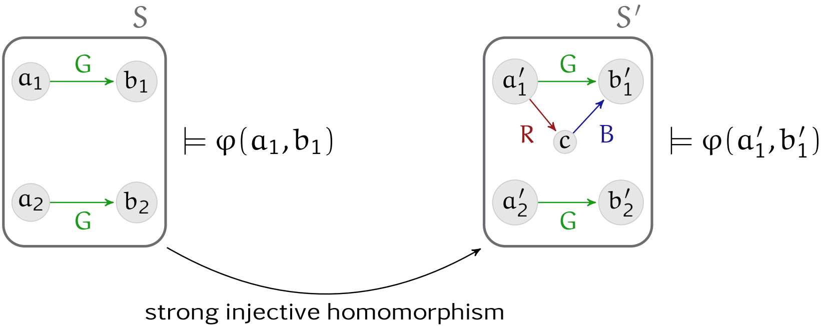

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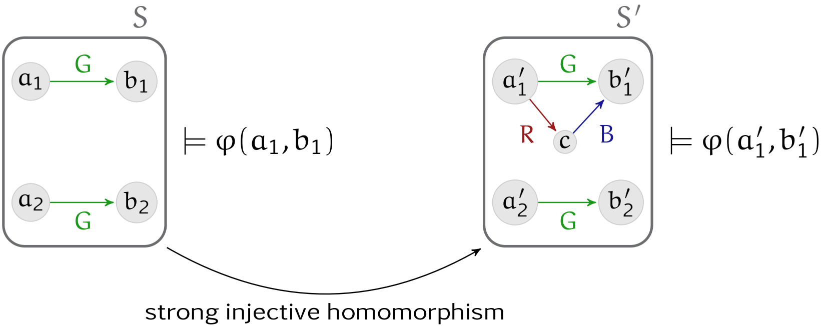
Formula $\varphi(x,y) = \exists z. x \xrightarrow{G} y \wedge \neg(y \xrightarrow{R} z) \in \exists\mathrm{FO}$



Application: Preservation Theorems

logic $\mathcal{L}$	example	$\mathrm{hom}_{\mathcal{L}}$
$\exists$FO	$\exists z. x \xrightarrow{G} y \wedge \neg(y \xrightarrow{R} z)$	strong injective
$\exists$FO$^+(\neq)$	$\exists y y'. x \xrightarrow{R} y \wedge y' \xrightarrow{B} z \wedge y \neq y'$	injective
$\exists$FO$^+$	$\exists y. x \xrightarrow{G} y$	all

Fact

If $\psi \in \mathcal{L}$, $h \in \mathrm{hom}_{\mathcal{L}}$, *and* $D \models \psi(\mathbf{x})$, *then* $h(D) \models \psi(h(\mathbf{x}))$.

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$\exists$FO^{+}	$\exists y. x \xrightarrow{G} y$	all

Definition

$D \leqslant_{\mathcal{L}} D'$ if $\exists h \in \hom_{\mathcal{L}}$ s.t. $D' = h(D)$.

Application: Preservation Theorems

logic $\mathcal{L}$	example	$\hom_{\mathcal{L}}$
$\exists$FO	$\exists z. x \xrightarrow{G} y \wedge \neg(y \xrightarrow{R} z)$	strong injective
$\exists$FO$^+(\neq)$	$\exists y y'. x \xrightarrow{R} y \wedge y' \xrightarrow{B} z \wedge y \neq y'$	injective
$\exists$FO$^+$	$\exists y. x \xrightarrow{G} y$	all

Over arbitrary structures

Theorem (Łoś, Lyndon, Tarski)

If φ is an FO-sentence s.t. $[\![\varphi]\!]$ is upwards-closed for $\leqslant_{\mathcal{L}}$, then there exists $\psi \in \mathcal{L}$ with $[\![\varphi]\!] = [\![\psi]\!]$.

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Over finite (relational) structures?

APPLICATION: PRESERVATION THEOREMS

logic $\mathcal{L}$	example	$\hom_{\mathcal{L}}$
$\exists$FO	no [Tait 1959]	strong injective
$\exists\text{FO}^{+}(\neq)$	no [Ajtai & Gurevich 1994]	injective
$\exists\text{FO}^{+}$	yes [Rossman 2008]	all

OVER FINITE (RELATIONAL) STRUCTURES?

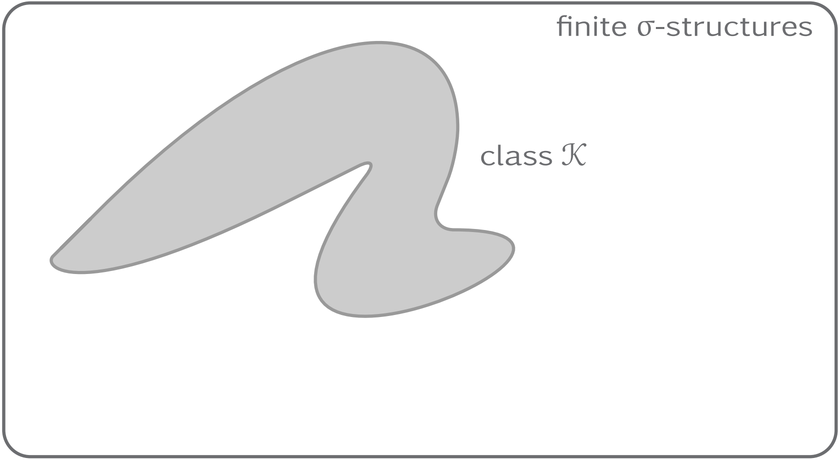
Application: Preservation Theorems

Over finite (relational) structures?

finite σ-structures

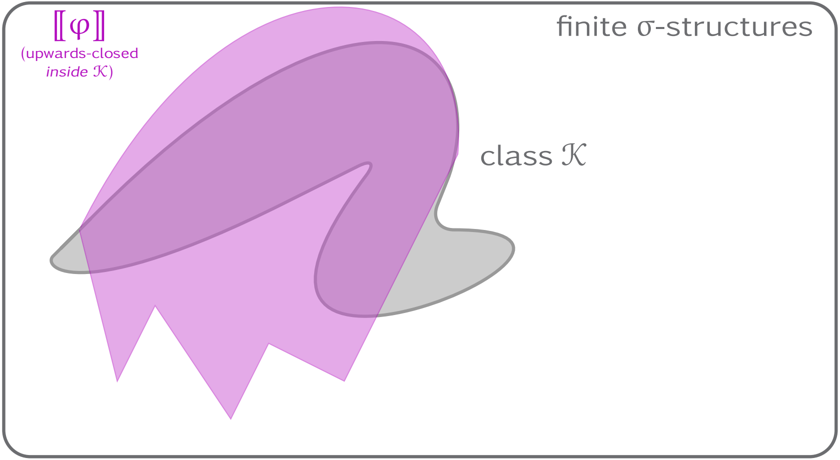
Application: Preservation Theorems

Over finite (relational) structures?



Application: Preservation Theorems

Over finite (relational) structures?



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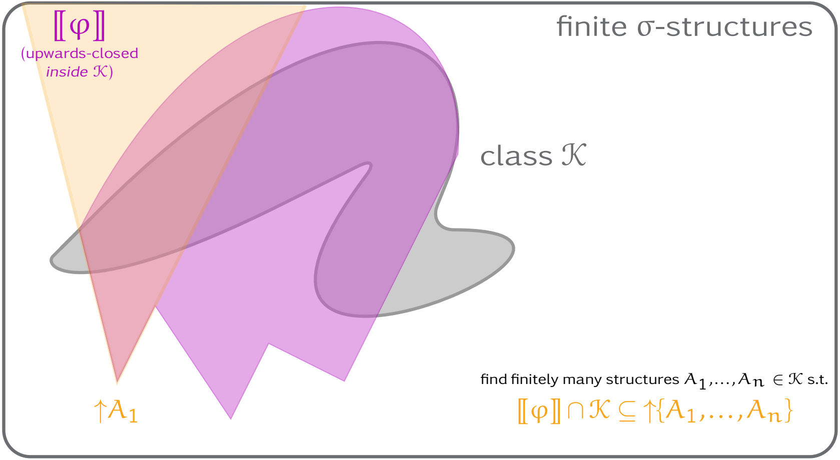

find finitely many structures $A_1, \dots, A_n \in \mathcal{K}$ s.t.

$$[\![\varphi]\!] \cap \mathcal{K} \subseteq \uparrow\{A_1, \dots, A_n\}$$

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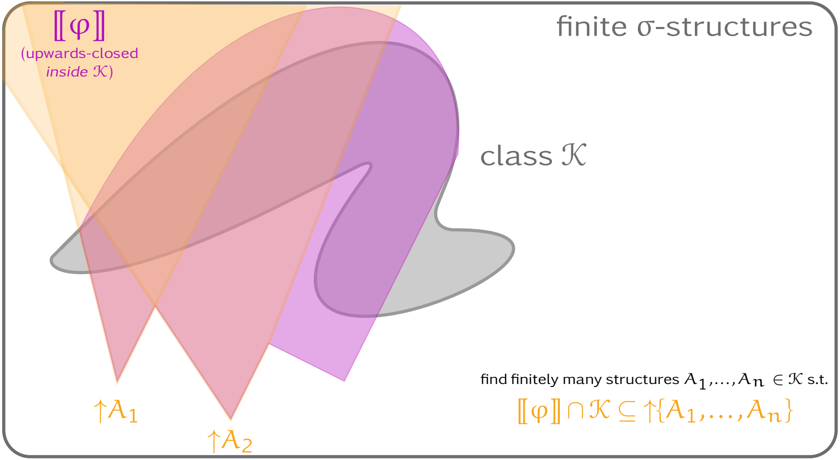

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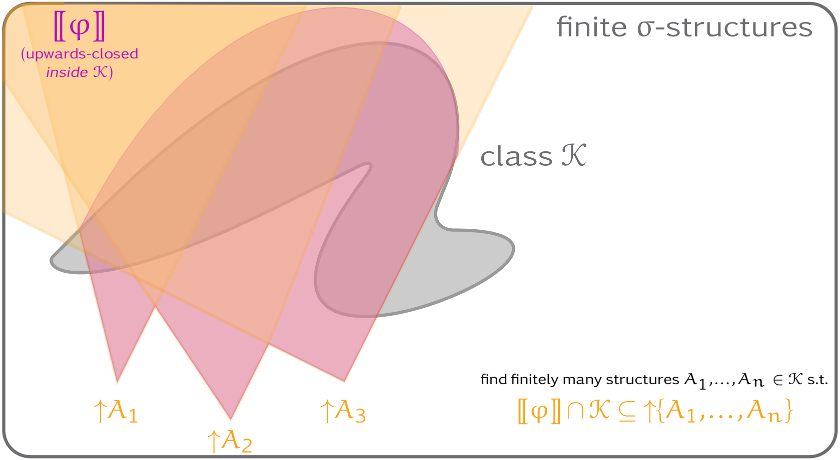

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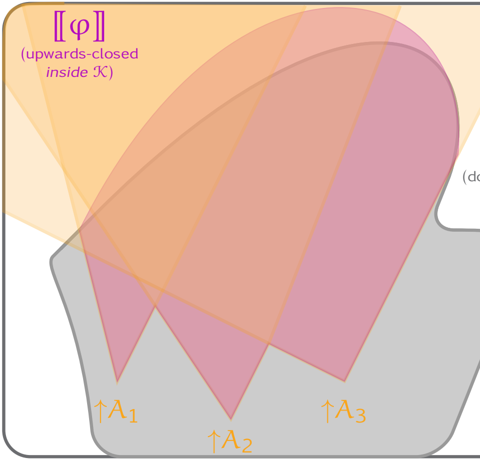

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Application: Preservation Theorems

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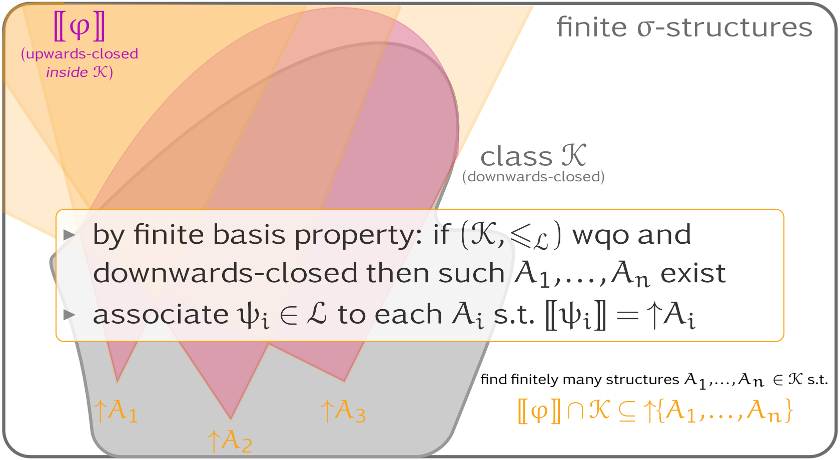
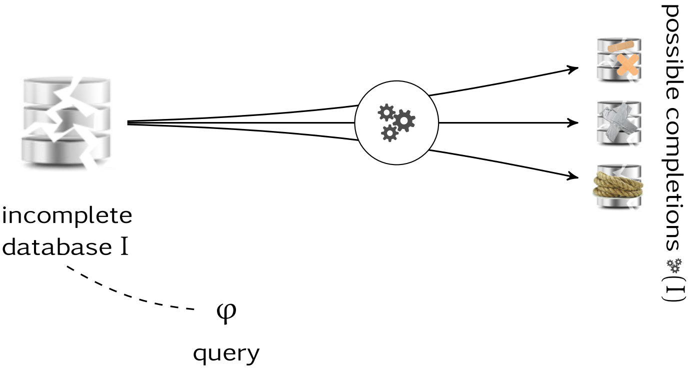


- by finite basis property: if $(\mathcal{K}, \leqslant_{\mathcal{L}})$ wqo and downwards-closed then such $A_1,\ldots,A_n$ exist
- associate $\psi_i \in \mathcal{L}$ to each A_i s.t. $[\![\psi_i]\!] = \uparrow A_i$

Application: Certain Answers



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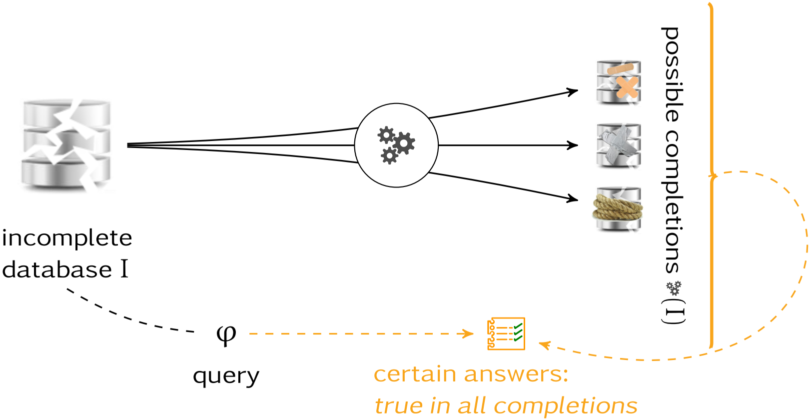
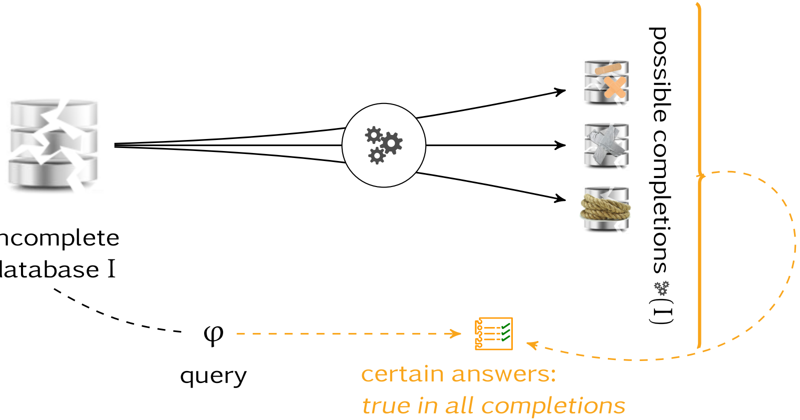
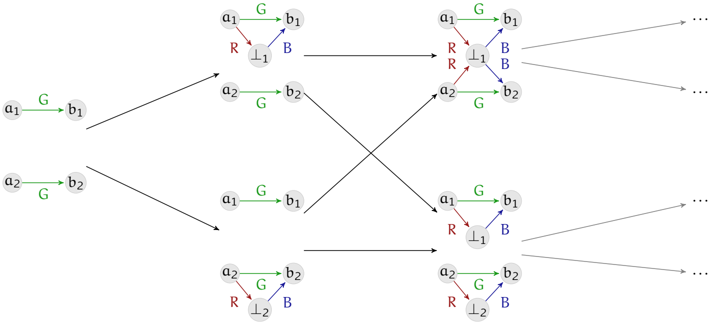


Application: Certain Answers



$$\mathrm{certain}_{I}(\varphi) = \bigcap_{D \in \text{⚙}(I)} \{\mathbf{x} \mid D \models \varphi(\mathbf{x})\}$$

Chase of $x \xrightarrow{G} z \implies \exists y\,.\, x \xrightarrow{R} y \wedge y \xrightarrow{B} z$ for $\varphi \in \exists\mathrm{FO}^+(\neq)$



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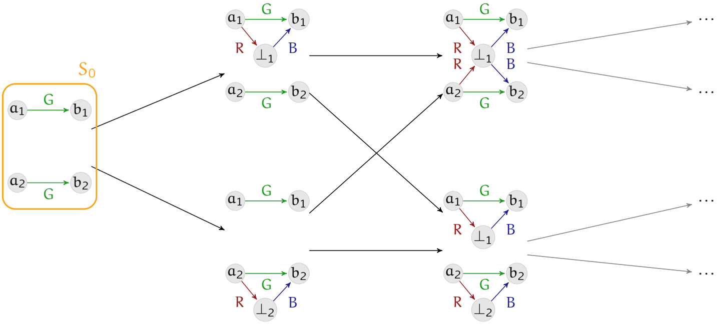

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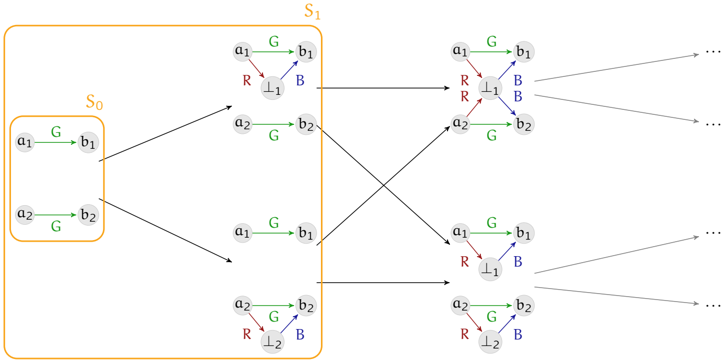

$\uparrow S_0 \quad \subseteq \quad \uparrow S_1$

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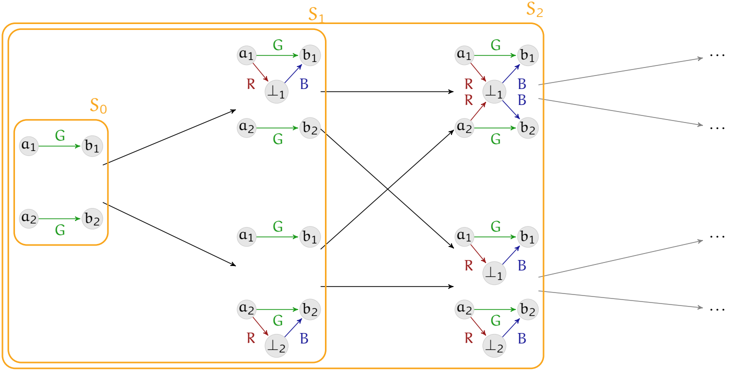

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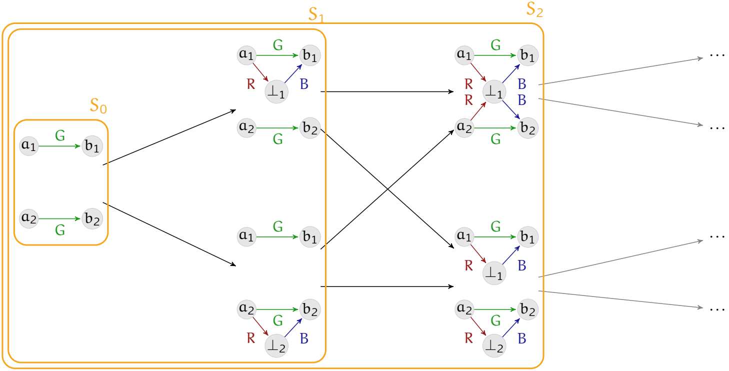

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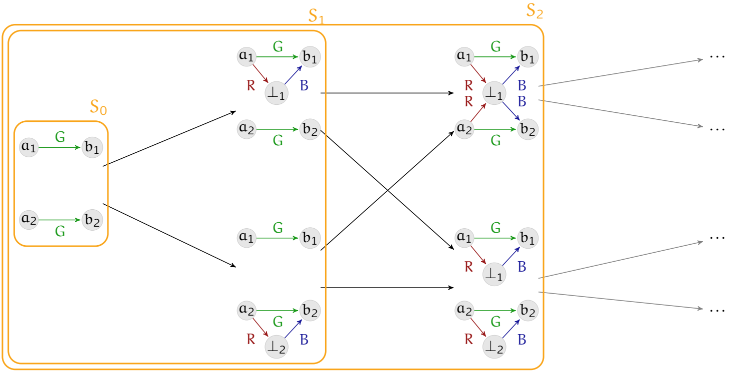

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- over a wqo: by ascending chain condition, $\uparrow S_0 \subseteq \uparrow S_1 \subseteq \cdots$ always stabilises to $\uparrow S_\star$
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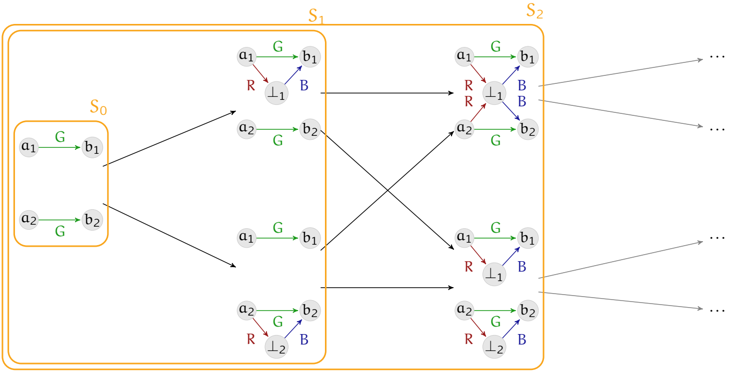

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Questions?