The tree property and its strengthenings

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Mitchell; Silver (1972): $\mathrm{TP}_{\aleph_2}$ is equiconsistent with a weakly compact cardinal.

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  - κ is *supercompact* if there is an elementary embedding as above, but M can be chosen to be "arbitrarily close" to V.

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  More on this later.

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A key point: like TP, the strong tree property and ITP can also hold at successor cardinals.

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What about ITP at successors of singular cardinals?

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- Unger / Fontanella, 2013: In this model actually $\mathrm{ITP}_{\aleph_n}$ holds for all $n > 1$.

What about ITP at successors of singular cardinals?

An immediate difficulty: no elementary embedding with such critical point.

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At successors of singulars, this does not quite work.

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The main obstacle with implementing Magidor-Shelah's strategy: the measures on say κ_0 and κ_{n+1} do not cohere.

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- do a careful interplay between the various normal measures.
- Finally: pick $\bar{\alpha} < \mu$ above which all of the above branches split. Show that one of them must be ineffable by partitioning a stationary subset of μ into κ_n many sets.

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- Use a model of Neeman for $TP_{\aleph_{\omega+1}}$, '12.
- Do Laver preparation, then force with $\mathbb{C} := \prod \mathrm{Col}(\kappa_n, <\kappa_{n+1})$.
- For each $\tau < \kappa_0$ of the form $\tau = \delta^+$ with $\mathrm{cf}(\delta) = \omega$, let $\mathbb{L}_\tau := \mathrm{Col}(\omega, \delta) \times \mathrm{Col}(\tau^+, <\kappa)$.

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- Here each κ_n becomes $\aleph_{n+3}$ and μ becomes $\aleph_{\omega+1}$.

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- Define a *system of branches* $\langle b_{\delta,\eta} \mid \delta < \kappa_n, \eta \in I \rangle$ through the system of the names of the lists.

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Key point: at all times have to consider all possible branches thought all possible names for lists.

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In particular, SCH fails at $\lambda^{+\omega}$.

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- pull back the branch to the right model.

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A key feature: cannot lift over the Prikry. So, we must work with names of lists.

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The key feature: Use interleaved collapses in the Prikry forcing. We need more involved lifting arguments and branch preservation lemmas.

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