About the existence of products of primes in $I\Delta_0$

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Contents

Bertrand's postulate

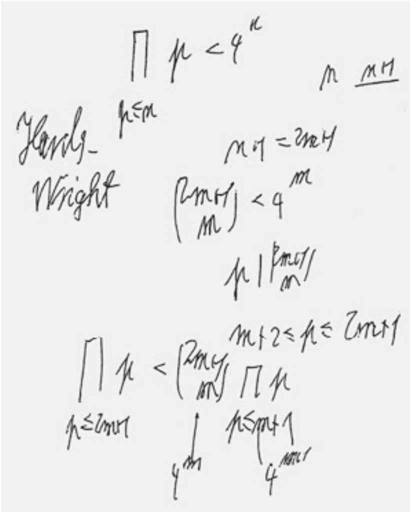
For every $n \geq 1$, there is some prime number p with $n < p \leq 2n$.

Erdos' proof of Bertrand's postulate

- Upper bound for the product of primes. For all real $x \geq 2$:

$$\prod_{p \text{ prime},\ p \leq x} p \leq 4^{x-1}$$

(It is enough to prove it for x=the largest prime $q \leq x$. The proof uses induction and the properties of central binomial coefficient).



$$\prod_{p \text{ prime},\, p\le 2m+1} p = \prod_{p \text{ prime},\, p\le m+1} p \cdot \prod_{p \text{ prime},\, m+1<p\le 2m+1} p \le 4^m \binom{2m+1}{m} \le 4^m 2^{2m} = 4^{2m}.$$

Lower bounds for the product of primes.

Legendre's theorem: The number $n!$ contains the prime factor p exactly

$$\sum_{k\geq 1,\, p^k\leq n} \left\lfloor \frac{n}{p^k} \right\rfloor$$

times.
The power of the prime factor p in $\binom{2n}{n} = \frac{(2n)!}{n!n!}$:

$$\sum_{k\geq 1,\, p^k\leq 2n} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor \right) \leq \sum_{k\geq 1,\, p^k\leq 2n} 1 \leq \max\{r : p^r \leq 2n\}.$$

The key fact of Erdos' proof!

Primes $p > \sqrt{2n}$ appear at most once in $\binom{2n}{n}$. If the prime factor p belongs to $\frac{2}{3}n < p \leq n$ and $n \geq 3$ then

$$\sum_{k \geq 1, p^k \leq 2n} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right) = 0$$

Divide the primes factors of $\binom{2n}{n}$ to "very small"($p \leq \sqrt{2n}$), "small"($\sqrt{2n} < p \leq \frac{2}{3}n$) and "large"($n < p \leq 2n$)

$$\frac{4^n}{2n} \leq \binom{2n}{n} \leq \prod_{p \text{ prime},\ p \leq \sqrt{2n}} p^{\text{some } k} \cdot \prod_{p \text{ prime},\ \sqrt{2n} < p \leq \frac{2}{3}n} p \cdot \prod_{p \text{ prime},\ n < p \leq 2n} p$$

$$\frac{4^n}{2n} \leq (2n)^{\sqrt{2n}} \cdot (4^{\frac{2}{3}n-1}) \cdot \prod_{p \text{ prime}, \; n<p\leq 2n} p$$

$$\frac{4^{n-\frac{2}{3}+1}}{(2n)^{1+\sqrt{2n}}} \leq \prod_{p \text{ prime}, \; n<p\leq 2n} p$$

But,

$$\frac{4^{\frac{n}{3}+1}}{(2n)^{1+\sqrt{2n}}} \geq 2^{\frac{n}{2}+2+\frac{1}{12}} \Leftrightarrow 2^{\frac{1}{12}(2n-1)} \geq (2n)^{1+\sqrt{2n}}$$

The surprise!

$$2n-1=(\sqrt{2n}-1)(\sqrt{2n}+1)\Rightarrow$$

$$2^{\frac{1}{12}(2n-1)}\geq(2n)^{1+\sqrt{2n}}\Leftrightarrow 2^{\sqrt{2n}-1}\geq(2n)^{12}\Leftrightarrow\sqrt{2n}-1\geq 12\log_2(2n)$$

This is true for $2n>(\frac{24}{\log 2})^2$ i.e. $n\geq 600$.

Chebyshev's Theorem:

$$2^{\frac{n}{2}+\frac{25}{12}}\leq\prod_{p\text{ prime},\ p\leq 2n}p\leq 4^{2n-1}$$

Question

Can we repeat the proof of (a) Betrand's postulate and (b) Chebyshev's result in $I\Delta_0$?

For (a) the answer is YES in the system $I\Delta_0 + PHP(\Delta_0)$.
A. R. Woods: *Some problems in logic and number theory and their connections*.
For (b) the answer is YES, in the system $I\Delta_0$.
P. D' Aquino: *Exponentiation and Fragments of Arithmetic*.

$\log_a^+(x), 1 \leq x < 4$

$\log x = \int_1^x \frac{1}{t} dt$. Fix $a \in M$ and $n \in \mathbb{N}$. For $1 \leq x < 4$ we define

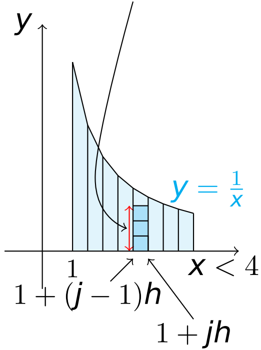
$$log_a^+(x) = \sum_{1 \leq j \leq [(x-1)/h]} \left(h^2 \left[\frac{1}{(1+jh)h} \right] \right)$$

where $h = \frac{1}{2^k} \in M$ is such that

$$h \leq \frac{1}{[\log_2 a]^n} < 2h$$

$log_a^+(x)$ = the area covered by the small squares with sides of length h under the graph of $y = \frac{1}{x}$

the number of h^2 squares entirely under the graph: $\left[\frac{1}{(1+jh)h}\right]$



Expanding the definition for all non negative rational numbers $x \geq 1$

$x = 2^{\log(x)} \frac{x}{2^{\log(x)}} \Rightarrow$

$$\log^*(x) = \log(x) \log_a^+(2) + \log_a^+\left(\frac{x}{2^{\log(x)}}\right).$$

From now on, we simply write $\log x$ instead of $[\log_2 x]$.

Properties of $\log *(x)$

- $\log^*$ is a step function. There are only $\leq 2^k(\log(x)+1)$ steps!
- There is a constant $K \in \mathbb{N}$ independent of a and x: $|\log^*(xy) - (\log^*(x) + \log^*(y))| < K \log^{-n}(a) < K(2h)$.
- For all $x \geq 2^k$: $\log^*(x+1) - \log^*(x) \leq 2^{-k} \leq \log^{-n}(a)$
- For all $x \geq \log^{2n}(a)$, $\sum_{1 \leq i \leq x} \log^* i = x \log^* x - x + O(\frac{x \log(x)}{\log^n(a)})$.

The first Chebyshev function

Let us suppose that $t = \prod_{p \text{ prime},\ 1 \le p \le x} p$ exists. We need good bounds for

$$\log^* t = \log^*\Big(\prod_{p \text{ prime},\ 1\le p\le x} p\Big).$$

$$\theta^*(x) = \sum_{1\le p\le x,\ p \text{ prime}} \log^*(p)$$

$$|\log^*(t) - \theta^*(x)| < \frac{K\#(1,x)}{\log^n(a)},$$

where $\#(n,m)$ denotes the number of primes in the interval $[n,m]$.

An approximation of $\theta(x)$

$$??\theta^*(x) = x + O(\frac{x}{\log^*(x)})$$

Results: For all x sufficient large we have:

$$\theta^*(x) = \sum_{p \leq x,\ p \text{ prime}} \log^* p \geq \frac{x}{4}.$$

Taking $n = 6, a = x$:

$$\log^*(t) = \theta^*(x) + O(\frac{\#(1,x)}{\log^6(x)}) \geq \frac{x}{4} - \frac{x}{8} \geq \frac{x}{8}$$

So 2^x exists!

The converse direction

There is some $C \in \mathbb{N}$, such that for each $r \geq 2$ the product

$$\prod_{p \text{ prime},\ 1 \leq p \leq C \log(r)} p$$

exists in M. Taking 2^x at place of r we take the result!
Why $\#(n,m)$ can be defined in M by a bounded formula? Answer: Define

$$check(i) = \begin{cases} 2, & if\, prime(i) \& n \leq i \leq m \\ 1, & otherwise \end{cases}$$

and

$$\#(n,m) = z \text{ iff } 2^z = \prod_{n \leq i \leq m} check(i)$$

The Ω hierarchy in relation to products of primes.

$$I\Delta_0 \vdash \forall m \geq 2 \forall z \geq 2 \Big(\prod_{p \text{ prime},\ 1 \leq p \leq z \log(m)} p \text{ exists} \leftrightarrow m^z \text{ exists} \Big)$$

Ω_1 denotes the axiom expressing "superpolynomial" power i.e.
$\Omega_1 : \forall x \forall y \exists z (x^{[log_2 y]} = z)$.

$$I\Delta_0 \vdash \forall m \left(\prod_{p \text{ prime},\ 1 \leq p \leq \log^2(m)} p \text{ exists} \right) \leftrightarrow \Omega_1$$

Number of consecutive primes

$$I\Delta_0 \vdash (\forall r > 1)(2^r \text{ exists } \rightarrow (\exists m)(\#(1, m) = r) \rightarrow 2^r \text{ exists })$$

Ευχαριστώ!