A Hales–Jewett type property of finite solvable groups

Miltiadis Karamanlis
kararemilt@gmail.com
(joint work with Vassilis Kanellopoulos)

National Technical University of Athens

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Ramsey sets

Some notation

- $\mathbb{N} = \{1, 2, \ldots\}$, $[r] = \{1, \ldots, r\}$, $|X|$ the cardinality of X.
- By an r-coloring of X, we mean any function $c : X \to [r]$.
- Given $c : X \to [r]$ we call $A \subset X$ monochromatic if c is constant in A.

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Definition

A finite set $X \subset \mathbb{R}^m$ is called Ramsey, if for every number of colors $r \in \mathbb{N}$, there exists a dimension $N \in \mathbb{N}$, such that for every r-coloring $c : \mathbb{R}^N \to [r]$, there exists an isometric copy $X' \subset \mathbb{R}^N$ of X which is monochromatic.

Previous results

1973 P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, *Euclidean Ramsey theorems*

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1991 I. Kříž, *Permutation groups in Euclidean Ramsey theory*

Two rival conjectures

Graham: All spherical sets must be are Ramsey!

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- I. Leader P. Russel M. Walters Transitive sets in Euclidean Ramsey Theory

LRW: Subtransitive sets must be Ramsey!

Variable words

Given:

- A finite set X (The alphabet)
- and $v \notin X$ (The variable)

...we call the elements of X^n as (words) and we call W a variable word over X of length $n \in \mathbb{N}$ if $W \in (X \cup \{v\})^n$ and v "appears" at last once.

Substitution in variable words

Given

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$$\begin{array}{llllll} W & = v & 1 & 2 & v & v \\ W(1) & = 1 & 1 & 2 & 1 & 1 \\ W(2) & = 2 & 1 & 2 & 2 & 2 \\ W(3) & = 3 & 1 & 2 & 3 & 3 \end{array}$$

Hales-Jewett theorem

Combinatorial Lines:

$$L = \{W(x) : x \in X\}$$

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Theorem (Hales-Jewett [1963])

For every finite alphabet X and any number of colors r, there exists $N = N(|X|, r) \in \mathbb{N}$, such that for every r-coloring of X^N, there exists a monochromatic combinatorial line.

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A multidimensional version also holds

H-variable words over X

Given:

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The degree d of W is $d = \sum |F_h|$, W is uniform if $|F_h| = |F_{h'}|$.

Substitution in H-variable words

Given:

- an H-variable word W over the alphabet X,
- a letter $x \in X$
- and a group action $\theta : G \times X \to X$ (i.e $\theta(e, x) = x$ and $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$)

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Conjecture

Conjecture (I.Leader, P. Russell, M. Walters)

For every finite group G and every positive integer r, there exist integers d and N such that for every r-coloring of G^N there exist a subset H of G and W, an H-variable word over G of degree d, such that:

$$\{W(g) : g \in G\}$$

is monochromatic.

...The action is assumed to be the left multiplication $\theta(h, g) = hg$.

Conjecture and transitive sets

Proposition

If Conjecture holds, then all transitive sets are Ramsey.

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Let $X \subset \mathbb{R}^m$ and G the symmetry group of X:

$$G = \{g \in X^X : \|gx_1 - gx_2\| = \|x_1 - x_2\|\}$$

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- Given r, find d and N as in Conjecture for G.
- Let $c : \mathbb{R}^{mN} \to [r]$.
- $c'(g_1 \ldots, g_N) = c\left(\frac{1}{\sqrt{d}}(g_1 x_0, \ldots, g_N x_0)\right)$

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- $c'(g_1 \ldots, g_N) = c\left(\frac{1}{\sqrt{d}}(g_1 x_0, \ldots, g_N x_0)\right)$
- Find c'-monochromatic W, (over G of degree d).
- $\left\{\frac{1}{\sqrt{d}}(W(e)_1 x, \ldots, W(e)_N x) : x \in X\right\}$ is a monochromatic copy of X.

□

UHJP

Definition (Uniform Hales-Jewett property)

We say that G *has the d-UHJP*, if for every $r \in \mathbb{N}$ there exists $N = N(G, d, r) \in \mathbb{N}$, such that for every r-coloring of G^N, there exists a d-uniform G-variable word over G of length N, such that the set $\{W(g) : g \in G\}$ is monochromatic.

...The action is assumed to be the left multiplication $\theta(h, g) = hg$.

Solvable groups

Recall that a finite group G is *solvable*, if it has a subnormal series $\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ such that G_i/G_{i-1} is cyclic for all $i \in [n]$.

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For a subnormal series as above, let p_i be the order of the factor group G_i/G_{i-1}. The number

$$\prod_{i=1}^{n} p_i^{(p_i-1)\prod_{j>i} p_j} = p_1^{(p_1-1)\prod_{j=2}^{n} p_j} p_2^{(p_2-1)\prod_{j=3}^{n} p_j} \dots p_n^{p_n-1} \tag{1}$$

will be called a *HJ-degree* of G.

Solvable groups have the UHJP

Theorem

All solvable groups G have the d-UHJP for every HJ-degree d of G.

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Corollary

Conjecture holds for Solvable groups.

UHJP revised

Definition (Uniform Hales-Jewett property revised)

Let $\theta : G \times X \to X$ be a group action. Also let H be a subgroup of G, E be an equivalence relation on X and $d \in \mathbb{N}$. We will say that (H, X) has the (θ, E, d)-UHJP, if for every $r \in \mathbb{N}$, there exists $N = N(H, X, \theta, E, d, r) \in \mathbb{N}$ such that for every r-coloring of X^N there exists a d-uniform H-variable word W over X of length N, such that for every $x \in X$ the set $\{W_\theta(x') : x' E x\}$ is monochromatic.

Theorem 2

Recall that:
$\theta : G \times X \to X$ induces $E_G \subset X \times X$:

$$xE_G y \Leftrightarrow x \in \{\theta(g, y) : g \in G\}$$

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Theorem

Let G be a solvable group, d a HJ-degree of G, X a finite set, and $\theta : G \times X \to X$ an action with p orbits. Then (G, X) has the (θ, E_G, d^p)-UHJP.

One dimensional regularity implies multidimensional

Proposition

Suppose that (H, X) has the (θ, E, d)-UHJP. Then for every $r \in \mathbb{N}$ and every $n \in \mathbb{N}$ there exists N such that, for every r-coloring of X^N we can find a sequence $(W_i)_{i=1}^n$ of d-uniform words such that:

$$\text{If } x_i E y_i \text{ for every } i \in [n] \text{ then } c\left(\prod_{i=1}^{n} W_i(x_i)\right) = c\left(\prod_{i=1}^{n} W_i(y_i)\right)$$

Cyclic groups have the UHJP

Proposition

Every cyclic group C of order p has the p^{p-1}-UHJP.

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Proof.

General inductive step:

- Assume we can find d uniform W s.t $c(W(e)) = \cdots = c(W(t^k))$

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General inductive step:

- Assume we can find d uniform W s.t $c(W(e)) = \cdots = c(W(t^k))$
- Find a large enough "space" which coordinate-wise has this regularity.

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- Assume we can find d uniform W s.t $c(W(e)) = \cdots = c(W(t^k))$
- Find a large enough "space" which coordinate-wise has this regularity.
- Use Ramsey's theorem in a clever way (due to Kříž) to find dp-uniform W s.t $c(W(e)) = \cdots = c(W(t^{k+1}))$.

□

UHJP behaves well with actions

Proposition

Let X a finite set, G a finite group and $\theta : G \times X \to X$ an action. If $H < G$ has the d-UHJP and $\theta : H \times X \to X$ has p orbits then (H, X) has the (θ, E_H, d^p)-UHJP.

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Let X a finite set, G a finite group and $\theta : G \times X \to X$ an action. If $H < G$ has the d-UHJP and $\theta : H \times X \to X$ has p orbits then (H, X) has the (θ, E_H, d^p)-UHJP.

Proof.

- Assume we can find W for which some of the orbits are monochromatic.
- Find large "space" which coordinate-wise has this regularity.
- Show that we can extent the regularity to hold for one more orbit.

□

Group extensions

Definition

Let G, K, H be groups. We say that G is an extension of K by H if:

$$H \xrightarrow{\iota} G \xrightarrow{\pi} K$$

- ι injective homomorphism, π surjective homomorphism.
- $\iota(H) = \ker\pi$.

UHJP is preserved under extensions

Proposition

Let H and K be finite groups and let G be an extension of K by H. If H has the d_H-UHJP and K has the d_K-UHJP then G has the $d_H^{|K|} \cdot d_K$-UHJP.

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Let H and K be finite groups and let G be an extension of K by H. If H has the d_H-UHJP and K has the d_K-UHJP then G has the $d_H^{|K|} \cdot d_K$-UHJP.

Proof.

The existence of the surjective homomorphism (the normality of H in G) plays a crucial role. □

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- Assume we have shown that G_{i-1} has the $d_{G_{i-1}}$-UHJP.
- G_i/G_{i-1} is cyclic hence has the $p_i^{p_i-1}$-UHJP.

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- G_i/G_{i-1} is cyclic hence has the $p_i^{p_i-1}$-UHJP.
- G_i is an extension of G_i/G_{i-1} by G_{i-1} hence has the $d_{G_{i-1}}p_i^{p_i-1}$-UHJP.

□

Interesting Questions

- Does Conjecture hold for all finite groups?
- What about the position of d?
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Thank you!