Integration in non-archimedean subanalytic geometry

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1. Subanalytic geometry

A **semianalytic subset** of $\mathbb{R}^n$ is locally given by finitely many equalities and inequalities of real analytic functions:

Examples:

- Semialgebraic sets are semianalytic.
- The graph of the sine function is semianalytic.

A **subanalytic subset** of $\mathbb{R}^n$ is locally given by a projection of a bounded semianalytic set:

A **globally subanalytic subset** of $\mathbb{R}^n$ is a set which is subanalytic in the ambient projective space.

Examples:

- Semialgebraic sets are globally subanalytic.
- Bounded subanalytic sets are globally subanalytic.
- The graph of the sine function is not globally subanalytic.

Central Fact:
The globally subanalytic sets form an **o-minimal structure** on the real field!

A **globally subanalytic function** is a function with globally subanalytic graph.

2. Model theoretic point of view

Definition:
A function $f:\mathbb{R}^n \to \mathbb{R}$ is **restricted analytic** if there is a power series p that converges on a neighbourhood of $[-1,1]^n$ such that

$$f(x) = \begin{cases} p(x), & \text{if } x \in [-1,1]^n, \\ 0, & \text{if } x \notin [-1,1]^n. \end{cases}$$

Let $\mathcal{L}_{\mathrm{an}}$ be the language of ordered rings augmented by symbols for every restricted analytic function; i.e.

$$\mathcal{L}_{\mathrm{an}} = \{+,\cdot,-,\leq,(f)_{\mathrm{restr.an.}}\}.$$

Let $\mathbb{R}_{\text{an}}$ be the natural $\mathcal{L}_{\text{an}}$-structure on the field of reals.

Then:

$$\text{Definable in } \mathbb{R}_{\text{an}} = \text{globally subanalytic.}$$

Let T_{an} be the $\mathcal{L}_{\text{an}}$-theory of $\mathbb{R}_{\text{an}}$. Then T_{an} has a natural axiomatization.

3. Non-archimedean models

Examples:

- Let Γ be an ordered abelian group that is divisible. Then the power series field $\mathbb{R}((t^\Gamma))$ can be made in a natural way into an $\mathcal{L}_{\mathrm{an}}$-structure. This structure is a model of T_{an}.

- The field $\mathbb{P}$ of (formal) **Puiseux series** over $\mathbb{R}$ in one variable

$$\left\{t^{-\frac{n}{q}}f\left(t^{\frac{1}{q}}\right)\ \middle|\ f\in\mathbb{R}[[t]], n\in\mathbb{N}_0 \text{ and } q\in\mathbb{N}\right\}$$

where $0<t\ll 1$ is a non-archimedean model of T_{an} with value group $\mathbb{Q}$.

- The field $\mathbb{P}_2$ of (formal) **Puiseux series** over $\mathbb{R}$ in two variables

$$\left\{s^{-\frac{m}{p}}t^{-\frac{n}{q}}f\left(s^{\frac{1}{p}},t^{\frac{1}{q}}\right)\ \middle|\ f\in\mathbb{R}[[s,t]], m,n\in\mathbb{N}_0 \text{ and } p,q\in\mathbb{N}\right\}$$

where $0<t\ll s\ll 1$ is a non-archimedean model of T_{an} with value group $\mathbb{Q}^2_{\mathrm{anlex}}$.

4. Motivation

For globally subanalytic sets and functions over the reals we have the powerful tools of analysis: differential calculus and integration theory.

For globally subanalytic sets and functions in the non-archimedean setting we have the powerful tool of differential calculus by o-minimality.

Now we want to establish an integration theory.

Obvious request:
The measure of an interval should be its length.

Warning:
Construction of the measure and the integral via **limits** is **not** possible.

Example:
For $n \in \mathbb{N}$ let $I_n := [n, t^{-1/n}] \subset \mathbb{P}$. Then $I_n \searrow \emptyset$ but the length of I_n equals $t^{-1/n} - n$ which is infinitely large.

Idea:
Use the powerful methods of **algebra and model theory**!

The **starting point** is the seminal work of Comte, Lion and Rolin:

Theorem (Comte, Lion, Rolin):
Let $n \in \mathbb{N}$ and let $p \in \mathbb{N}_0$. Let $A \subset \mathbb{R}^{p+n}$ be globally subanalytic. The following holds:

(1) The set
$$\mathrm{Fin}(A) := \left\{ t \in \mathbb{R}^p \mid \lambda_n(A_t) < +\infty \right\}$$
is globally subanalytic.

(2) The function
$$\mathrm{Fin}(A) \to \mathbb{R}_{\geq 0}, t \mapsto \lambda_n(A_t),$$
is given by a finite sum of finite products of globally subanalytic functions and logarithms of positive globally subanalytic functions.

Here λ_n denotes the usual Lebesgue measure on $\mathbb{R}^n$.

5. Construction

From now on we work under the following general assumption:

Let R be a model of T_{an} such that the value group $\Gamma = \Gamma_R$ has finite archimedean rank ℓ.

Note that $\ell = 0$ if and only if $R = \mathbb{R}$.

We denote by

- $\mathfrak{m}_R$ the set of infinitesimal elements of R,
- $\mathcal{I}_R$ the set of positive infinitely large elements of R.

5.1 Logarithms

The T_{an}-model R carries a **partial logarithm**

$$\log : \mathbb{R}_{>0} + \mathfrak{m}_R \to R, a + m \mapsto \log(a) + L\Big(\frac{m}{a}\Big),$$

where L denotes the logarithmic series.

Example:
In the case of the field $\mathbb{P}$ of Puiseux series let

$$f = a + bt^q + \textit{lower terms} \in \mathbb{R}_{>0} + \mathfrak{m}_{\mathbb{P}}.$$

(i.e. $a \in \mathbb{R}_{>0}, b \in \mathbb{R}, q \in \mathbb{Q}_{>0}$).
Then

$$\begin{aligned} \log(f) &= \log\left(a(1 + \tfrac{b}{a}t^q + \ldots)\right) \\ &= \log(a) + \log(1 + \tfrac{b}{a}t^q + \ldots) \\ &= \log(a) + L(\tfrac{b}{a}t^q + \ldots). \end{aligned}$$

We want to extend the partial logarithm to a **global logarithm**.
We have to extend the range.

We denote by $R[X] = R[X_1, \ldots, X_\ell]$ the polynomial ring over R in ℓ variables. We order $R[X]$ by setting

$$1 \ll X_1 \ll \ldots \ll X_\ell \ll \mathcal{I}_R.$$

We construct a **global logarithm** $R_{>0} \to R[X]$ with good properties. The construction depends on algebraic choices. The rough idea is that one chooses representatives

$$1 \ll a_1 \ll \ldots \ll a_\ell$$

of the archimedean classes in a compatible way and sets

$$\log(a_1) := X_1, \ldots, \log(a_\ell) := X_\ell.$$

Example:
In the case of the field $\mathbb{P}$ of Puiseux series let

$$f = at^q + \textit{lower terms} \in \mathbb{P}_{>0}.$$

(i.e. $a \in \mathbb{R}_{>0}, q \in \mathbb{Q}$).
Then

$$\begin{aligned}\log(f) &= \log\left(at^q(1 + \textit{infinitesimal})\right)\\ &= \log(a) - qX + L(\textit{infinitesimal}).\end{aligned}$$

5.2 Embedding into tame models

The above construction is of algebraic nature. We want to go back into the model theoretic setting.

Let $\mathbb{R}_{\mathrm{an,exp}}$ be the structure obtained by expanding $\mathbb{R}_{\mathrm{an}}$ by the global exponential function on the reals. It is o-minimal.
Let $\mathcal{L}_{\mathrm{an,exp}}$ be the language $\mathcal{L}_{\mathrm{an}}$ augmented by a symbol for exponentiation and let $T_{\mathrm{an,exp}}$ be the theory of the natural $\mathcal{L}_{\mathrm{an,exp}}$-structure $\mathbb{R}_{\mathrm{an,exp}}$.

Given a model S of $T_{\mathrm{an,exp}}$ we denote its logarithm by $\log_S$.

Theorem:
Let $\log : R_{>0} \to R[X]$ be a global logarithm. Then there is a model S of $T_{\mathrm{an,exp}}$, an $\mathcal{L}_{\mathrm{an}}$-embedding $\rho : R \hookrightarrow S$ and a tuple $Y = (Y_1, \ldots, Y_\ell) \in S^\ell$ such that the following conditions hold.

(a) The homomorphism $\rho^Y : R[X] \to S$ which extends ρ and maps X to Y is an order preserving embedding.

(b) The diagramm

$$\begin{array}{ccc} R_{>0} & \xrightarrow{\log} & R[X] \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho^Y} \\ S_{>0} & \xrightarrow{\log_S} & S \end{array}$$

commutes.

Idea of the proof:
Fields of transseries.

5.3 Model theoretic transfer

Central idea:

Result on the reals with parameters

$$\longleftrightarrow$$

Result in non-standard structures.

This allows to transfer the parametric result of Comte, Lion and Rolin to non-archimedean models by the choice of a global logarithm and an embedding into a model of $T_{\mathrm{an,exp}}$.

5.4 Lebesgue integration theory

We obtain (after the choice of a global logarithmic) a **Lebesgue measure**

$$\Big\{\text{Globally subanalytic subsets of } R^n\Big\} \to R[X]_{\geq 0} \cup \{\infty\}$$

that is

- finitely additive,
- monotone,
- translation invariant,

and

- reflects elementary geometry,

and a **Lebesgue integral** with value in $R[X]$ of ‘integrable’ globally subanalytic functions on R^n such that the integral is

- linear, monotone

and the

- transformation formula,
- fundamental theorem of calculus,
- Lebesgue's theorem on dominated convergence,
- Fubini's theorem

can be established.