# Some model theory of elliptic functions

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# Disclaimer

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Model theory here is that of expansions of the real field. There are other interesting model theoretic points of view on elliptic functions, but I won't mention them.

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So we can reduce to studying $\wp$ on a fundamental domain.

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So, we first recall some results in the model theory of exponentiation.

# Exponentiation

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These results also hold for the expansion of the real field by exp restricted to the strip $\mathbb{R} \times [-i\pi, i\pi]$ in the complex plane. (We identify $\mathbb{C}$ with $\mathbb{R}^2$, so expanding the reals by a complex function means adding its real and imaginary parts.)

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The constants here are a potential issue for decidability, and it is not clear how to make Bianconi's proof effective.

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*Let* $\Omega = \mathbb{Z} + i\mathbb{Z}$, *and* $c_\Omega = \sqrt{\pi}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$. *Suppose that a certain analogue of Schanuel's conjecture,* $\wp_\Omega$ *in place of* exp, *holds. Then the theory of* $(\overline{\mathbb{R}}, \wp_\Omega|_{\mathcal{F}_\Omega}, c_\Omega)$ *is decidable.*

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# Outline of Macintyre's proof

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3. Use Step 2 and the transcendence conjecture to get decidability.

# Pfaffian functions

A sequence $f_1, \dots, f_l$ of analytic functions on an open set $U \subseteq \mathbb{R}^n$ is a pfaffian chain if there are polynomials $P_{i,j} \in \mathbb{R}[X_1, \dots, X_n, Y_1, \dots, Y_i]$ for $j = 1, \dots, n$ and $i = 1, \dots, l$ such that

$$\frac{\partial f_i}{\partial x_j}(x) = p_{i,j}(x, f_1(x), \dots, f_i(x)).$$

A function $f : U \to \mathbb{R}$ is pfaffian, with chain $f_1, \dots, f_l$, if there is a polynomial $p$ such that $f(x) = p(x, f_1(x), \dots, f_l(x))$. Say $f$ has length $l$, degree $(\max\{\deg p_{i,j}\}, \deg p)$.

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Macintyre asked, among several other questions, the following:

1. Is $T_W$ model complete?
2. Is $T_W$ decidable (relative to some transcendence conjecture)?

# Piecewise semipfaffian sets

A set $X \subseteq \mathbb{R}^n$ is piecewise semipfaffian if there are simple domains $U_i \subseteq \mathbb{R}^n$ for $i = 1, \ldots, L$ and pfaffian functions $f_{i,1}, \ldots, f_{i,m_i} : U_i \to \mathbb{R}$, with a common chain of length $r$ and degree $(\alpha, \beta)$ such that

$$X = \bigcup \{x \in U_i : f_{i,1} = \cdots = f_{i,m_i} = 0\}.$$

If $m_i \leq M$, we say that $X$ has complexity $(r, \alpha, \beta, n, L, M)$.

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*Suppose that $P$ is a polynomial in two variables, with complex coefficients, not identically zero and of total degree bounded by $L \geq 20$. Then on $\mathcal{F}_\Omega$ the function $P(z, \wp(z))$ has at most*

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I'm fairly confident that combining the uniform pfaffian definition with Foster's modification of Wilkie's proof of model completeness for restricted pfaffian functions and ideas of Macintyre's will show that $T_W$ is model complete (perhaps in a slightly extended language).

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Thank you!