Indefinitely Extensible Models and a Relative Infinite

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Introduction

Claim.

The concept of an actual infinite set is not necessary to do mathematics, the potential infinite suffices and avoids paradoxes.

The Path to this Goal.

Adopt model theory to use a potential infinite carrier set. The $\forall$-quantifier refers to a sufficiently large finite set.

Forerunners.

- Jan Mycielski (Local finite theories, JSL '86).
- Shaughan Lavine (Understanding the Infinite).

Introduction

Main Ideas.

- The potential infinite is a dynamic concept of infinity.
  - Actual infinity sees infinity as a *size*.
  - Potential infinity sees infinity as a *process*.
- No specific philosophical position required.
- However, it is a form of finitism.
- But: Concepts apply to other small-large distinctions instead of finite-infinite as well.
- Very often the potential infinite can be treated as if it is actual.
- No notion of computability involved.
- No change of proof theory (at least for FOL).
- For HOL use types, e.g. simple type theory, and regard a function as an extensible object.

The Relative Infinite.

Indefinitely Extensible and Indefinitely Large.

A better term than "potential infinite" is "indefinitely extensible" (cf. Dummett) — Extensibility is more fundamental than completion.
Based on indefinitely extensible it is possible to define "indefinitely large" or "sufficiently large" finite states — Completion is possible, but only temporary.

The Relative Infinite.

Infinity is a relative notion which has two parts:

- The indefinitely extensible set.
- Indefinitely large (finite) sizes within the indefinitely extensible set.

The Relative Infinite

Inside the Process, not Outside.

Every completion of an indefinitely extensible set/process is just another state inside the set/process (basis for a reflection principle).

Reference.

In a formula as $\forall \mathsf{x}_0 \exists \mathsf{x}_1 \Phi$, the variables x_0 and x_1 typically refer to different states.

Example Natural Numbers.

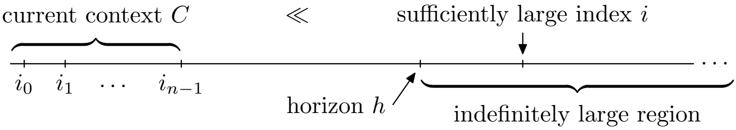
Instead of $\mathbb{N}$ use $(\mathbb{N}_i)_{i \in \mathbb{N}}$ with $\mathbb{N}_i = \{0, \ldots, i-1\}$.

- Concept "number of natural numbers" is indefinitely extensible (in Dummett's sense).
- ω is replaced by an indefinitely large finite number.

The Relative Infinite

Threefold Relativity of the Infinite.

- Not a single state $i \in \mathcal{I}$, but a region, e.g. $\{i' \in \mathcal{I} \mid i' \geq i\}$, the "indefinitely large region".
- The region depends on a context $C = (i_0, \ldots, i_{n-1}) \rightsquigarrow$ notion $C \ll i$ (or $i \gg C$), "i is indefinitely large relative to C".
- Not a single relation $\ll$, but several ones. They satisfy:
  - $C \ll i \leq i'$ implies $C \ll i'$,
  - $(i_0, \ldots, i_{k-1}) \ll i_k$ for all $k < n$.



First Order Logic

Model Theoretic Adoptions.

Classical first order predicate logic.

- Indefinitely extensible carrier set: $\mathcal{M} \rightsquigarrow \mathcal{M}_{\mathcal{I}} := (\mathcal{M}_i)_{i \in \mathcal{I}}$ with finite sets $\mathcal{M}_i$ and a directed index set $\mathcal{I}$.
- Assume a notion of an *indefinitely large size*, i.e., a relation $C \ll i$ with $C = (i_0, \dots, i_{n-1})$.
- $\models \Phi[\boldsymbol{a}] \rightsquigarrow \models_{\ll} \Phi[\boldsymbol{a} : C]$ with $\boldsymbol{a} = (a_0, \dots, a_{n-1})$, $a_k \in \mathcal{M}_{i_k}$ replacing the free variables $\mathrm{x}_0, \dots, \mathrm{x}_{\mathrm{n}-1}$ of Φ.
- Main change is the interpretation of the $\forall$-quantifier:

$$\models_{\ll} \forall \mathrm{x}_{\mathrm{n}} \Phi[\boldsymbol{a} : C] \;:\Longleftrightarrow\; \models_{\ll} \Phi[\boldsymbol{a}, b : C.i] \text{ for all } b \in \mathcal{M}_i$$

  for some (sufficiently large) index $i \gg C$.
- Interpretation of $\forall \mathrm{x}_{\mathrm{n}} \Phi$ is independent of the choice of $i \gg C$.

First Order Logic

Adequacy

Given a finite set $\mathcal{T}$ of formulas — language $\mathcal{L} = (\mathcal{L}_k)_{k\in\mathbb{N}}$ is also indefinitely extensible, i.e., $\mathcal{L}_k$ finite and $\mathcal{T} \subseteq \mathcal{L}_k$.

- $\models_{\ll}$ is adequate for $\mathcal{T}$ if $C \ll i$ implies that $\mathcal{M}_i$ contains all witnesses of valid existential quantified formulas (c.f. Löwenheim-Skolem theorem).
- Adequacy guarantees a sound and complete interpretation.
  - Direct proof via a Henkin model.
  - Indirect proof: Transformation of models $\mathcal{M}_\mathcal{I} \longleftrightarrow \mathcal{M}$, e.g. $\mathcal{M} = \bigcup_{i\in\mathcal{I}} \mathcal{M}_i$.
  - Completeness requires non standard models, e.g. Henkin model for PA.

First order logic

Further Metamathematical Properties.

- Compactness states the existence of a uniform model (follows immediately from Henkin construction).
- No unavoidable non-standard models.
  - Non-standard models are possible (e.g. Henkin model of PA), but
  - non-standard elements are introduced at some stage $i \in \mathcal{I}$.

Self Application.

Concepts are applicable to background model (meta level): Model of model theory uses indefinitely extensible sets, in particular: Index $\mathcal{I}$ is not actual infinite.

First order logic

Intuitionistic Logic

Concepts are applicable to Kripke models. Two index sets:

- Ontological states $i \in \mathcal{I}$ (as mentioned above for classical logic).
- Epistemological states $k \in \mathcal{K}$ of knowledge (as usual Kripke models).

Differences:

- $\mathcal{I}$ directed, $\mathcal{K}$ often "tree-like".
- Relation R_C^k at node k and state C satisfies:

$$R_C^k(\boldsymbol{a}) \iff R_{C'}^k(\boldsymbol{a}) \qquad \text{for } \boldsymbol{a} \in \mathcal{M}_C \cap \mathcal{M}_{C'}$$
$$R_C^k(\boldsymbol{a}) \Rightarrow R_C^{k'}(\boldsymbol{a}) \qquad \text{for } k \leq k' \text{ and } \boldsymbol{a} \in \mathcal{M}_C.$$

Possible: Knowledge increases on the same ontological context C.

Towards Higher Order Logic

This is work in progress. HOL is a bigger challenge than FOL.

New in HOL.

Infinite objects are approximated, too. No infinite functions as $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$, but approximations $f : (\mathbb{N}_i \to \mathbb{N}_j) \to \mathbb{N}_k$ with $\mathbb{N}_i = \{0, \ldots, i-1\}$ and $i, j, k \in \mathbb{N} \setminus \{0\}$.

Principle of finite support is automatically fulfilled.

Concepts which do not work.

These concepts already require a notion of infinity:

- Domain theory due to *direct completeness*.
- Hyperfinte type structure due to a *Fréchet product*.

Towards Higher Order Logic

Concepts which work.

Notions of a *system* like the direct or inverse system.

Factor Systems.

Factor systems generalize direct and inverse system, having both as extrem cases. Basic notion is

$$a' \stackrel{p}{\mapsto} a \ \text{ for } \ a' \in \mathcal{M}_{i'}, a \in \mathcal{M}_i, i' \geq i,$$

read as "a approximates a' at level i". No transitivity required (does not hold for logical relations), but property:

$$a'' \stackrel{p}{\mapsto} a', \ a'' \stackrel{p}{\mapsto} a \ \Rightarrow \ a' \stackrel{p}{\mapsto} a$$

for $a'' \in \mathcal{M}_{i''}$, $a' \in \mathcal{M}_{i'}$, $a \in \mathcal{M}_i$, $i'' \geq i' \geq i$. In most cases $\stackrel{p}{\mapsto}$ is a partial surjection.

Towards Higher Order Logic

PER-sets.

- PER-sets are limits of factor systems.
- They naturally have families of PERs (indexed over $\mathcal{I}$).
- These PERs replace equality (increasing finer equality).
- Partiality necessary, e.g., $id : \mathbb{N} \to \mathbb{N}$ has approximations $id_{i \to j} : \mathbb{N}_i \to \mathbb{N}_j$ only if $i \leq j$.
- The function space of PER-sets consists only of uniform continuous functions.

New Concept of Function.

- Functions are indefinitely increasing sets of assignments $a \mapsto f(a), a' \mapsto f(a'), \ldots$.

Open questions

Done so far.

- An interpretation with reflection principle for (classical and intuitionistic) FOL — submitted to the NDJFL.
- A model for simple type theory — will be submitted soon.

Open Questions for HOL.

- How to interpret λ-terms, or at least formulas for HOL?
- How are PER-sets related to the hereditarily total continuous functionals (Kleene-Kreisel functionals)?
- Does the new interpretation of a function affect properties/proofs about them?

Any questions?

Thank you for your attention.