Iterated forcing and Forcing Axioms

Mirna Džamonja, Tutorial 2

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This can get a little complicated because of two issues: preservation of the axioms and preservation of cardinals.

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Instead we need to deal with a forcing which is entirely in V_0 and consists of **names**.

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(2) $V[G] = \{\underset{\sim}{\tau}_G : \underset{\sim}{\tau} \in V\}$.

The forcing relation

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Suppose that $\varphi(x_0, x_1, \ldots x_{n-1})$ is a formula in the language of set theory $\mathcal{L} = \{\in\}$ and $\underset{\sim}{\tau}^0, \underset{\sim}{\tau}^1, \ldots \underset{\sim}{\tau}^{n-1}$ are $\mathbb{P}$-names, while $p \in \mathbb{P}$.

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Forcing Theorem, part 2 (1) For any φ and $\underset{\sim}{\tau}^0, \underset{\sim}{\tau}^1, \dots \underset{\sim}{\tau}^{n-1}$, V as above, $V[G] \models \varphi(\underset{\sim}{\tau}^0_G, \underset{\sim}{\tau}^1_G, \dots \underset{\sim}{\tau}^{n-1}_G)$ iff for some $p \in G$ we have $p \Vdash \varphi(\underset{\sim}{\tau}_0, \underset{\sim}{\tau}_1, \dots \underset{\sim}{\tau}_{n-1})$.

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(2) There is a relation $\Vdash^*$ definable in the ground model such that $p \Vdash \varphi$ iff $p \Vdash^* \varphi$.

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- $\{\gamma < \alpha : \neg(p \restriction \gamma \Vdash p(\gamma) = \underset{\sim}{\emptyset}_\gamma)\}$ is finite.

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It is possible to arrange the "bookkeeping" so that $\langle \underset{\sim}{R}_\alpha : \alpha < \omega_2 \rangle$ goes over all relevant names, i.e. any Souslin tree in the extension by P_{ω_2} is named at some stage as $\underset{\sim}{R}_\alpha$.

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For all said so far can use as reference ""Proper and Improper Forcing" by Shelah and for iterated forcing also the article "Iterated Forcing" by Jim Baumgartner.

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We have got a solution for you ....

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Using $MA + \neg CH$ set theorists and non-set theorists have proved a variety of consistency results, mostly about ω_1.

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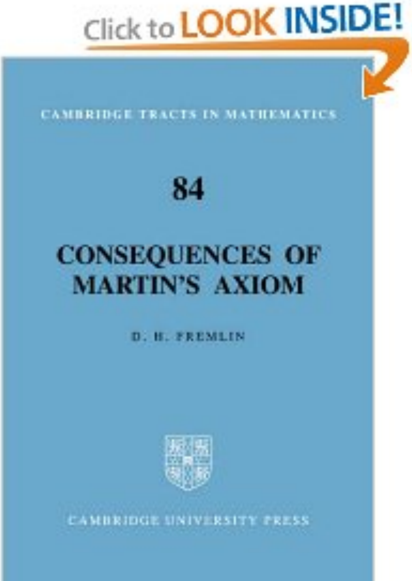

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(Baumgartner 1984) Modulo a supercompact cardinal, PFA is consistent.

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A posteriori, this is a lot like master conditions in large cardinal forcing.
Note ccc implies proper.

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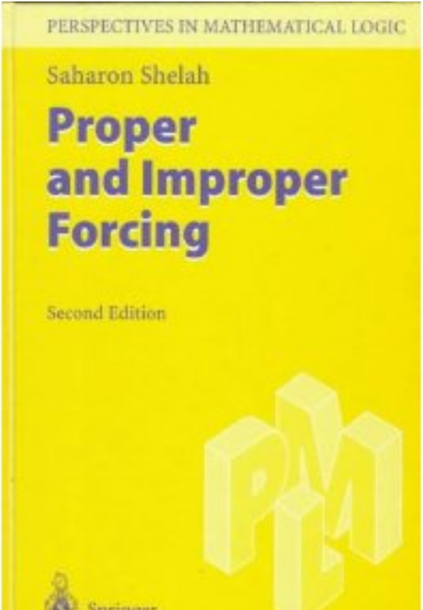

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Introduced the idea of *strongly proper* forcing.

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Note: to destroy a square "thread a club".

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Suppose that $\mathcal{M}_1, \mathcal{M}_2 \prec H_{\omega_2}$ are countable and let $\delta := \sup(M_1 \cap M_2)$. Then the set $\{\min(M_1 \setminus \lambda) \mid \lambda \in M_2,\ \delta < \lambda < \sup(M_1)\} \cup \{\min(M_1 \setminus \delta)\}$ is called the set of M_1*-fences for* M_2;

Definition

The forcing notion P is the set of conditions of the form $p := (\mathcal{F}_p, \ \mathcal{S}_p, \ \mathcal{O}_p, \ \mathcal{M}_p)$, where

(1) $\mathcal{F}_p : \mathrm{Lim}(\omega_2) \to \mathcal{P}(\omega_2)$, $|\mathcal{F}_p| < \omega$ and for all $\alpha \in \mathcal{D}_p := \mathrm{dom}(\mathcal{F}_p)$, $\mathcal{F}_p(\alpha)$ is a club $C_\alpha \subset \alpha$ whose order type is $< \omega_1$ if $\mathrm{cf}(\alpha) = \omega$ and which satisfies $C_\alpha \in \{E_\alpha \setminus \beta \mid \beta \in \mathcal{D}_p \cap \alpha\}$ if $\mathrm{cf}(\alpha) = \omega_1$;

(2) $\mathcal{S}_p \subset \mathcal{D}_p$ and $\alpha \in \mathcal{S}_p$ for every $\alpha \in \mathcal{D}_p$ with $\mathrm{cf}(\alpha) = \omega_1$;

(3) $\mathcal{M}_p$ is a finite set of sets $M \cap \omega_2$ for some ocountable $\mathcal{M}[M] \prec H_\theta$, and $\sup(M) \in \mathcal{S}_p$ for every $M \in \mathcal{M}_p$;

(4) for every $\alpha \neq \beta \in \mathcal{D}_p$, if $\mu \in \mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta)$ then $C_\alpha \cap \mu = C_\beta \cap \mu$;

(5) if $\alpha \in \mathcal{D}_p$ and $\sigma \in \mathcal{S}_p \cap \alpha$, then $C_\alpha \cap \sigma$ is a finite set;

(6) for all $\alpha \in \mathcal{D}_p$ and $M \in \mathcal{M}_p$:

(a) if $\alpha \in M$ then $C_\alpha \in \mathcal{M}[M]$,

(b) if $\alpha \notin M$ is such that $\alpha < \sup(M)$, or if $\alpha \in M$ is such that $\sup(M \cap \alpha) < \alpha$, then $\min(M \setminus \alpha) \in \mathcal{S}_p$ and $\sup(M \cap \alpha) \in \mathcal{D}_p$[1],

[1] Note that if $\alpha \in M$ then $\sup(M \cap \alpha) < \alpha$ iff $\mathrm{cf}(\alpha) = \omega_1$.

Definition

(c) if $\alpha \notin M$, $\sup(M \cap \alpha) < \alpha < \sup(M)$ and there is no $\beta \in \mathcal{D}_p \setminus (\alpha + 1)$, such that $\alpha \in \mathrm{Lim}(C_\beta)$, then $C_\alpha \cap \sup(M \cap \alpha)$ is a finite set,
(d) if $\alpha \notin M$, $\sup(M \cap \alpha) = \alpha$ and there is no $\beta \in \mathcal{D}_p \setminus (\alpha + 1)$, such that $\alpha \in \mathrm{Lim}(C_\beta)$, then C_α is some cofinal sequence in α of length ω;
(7) $\mathcal{O}_p$ is a finite set of half open nonempty intervals $(\beta', \beta] \subset \omega_2$ such that $\mathcal{D}_p \cap \bigcup \mathcal{O}_p = \emptyset$;
(8) if $(\beta', \beta] \in \mathcal{O}_p$ and $M \in \mathcal{M}_p$ then either $(\beta', \beta] \in \mathcal{M}$ or $(\beta', \beta] \cap \mathcal{M} = \emptyset$;
(9) if $M_1, M_2 \in \mathcal{M}_p$ then they are compatible, and the M_1-fence for M_2 is a subset of $\mathcal{S}_p$.

For $p, q \in P$ define

$$p \leq q \overset{\text{def}}{\iff} \mathcal{F}_p \subset \mathcal{F}_q,\ \mathcal{S}_p \subset \mathcal{S}_q,\ \mathcal{O}_p \subset \mathcal{O}_q,\ \mathcal{M}_p \subset \mathcal{M}_q.$$

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The conditions in the iteration are a mixture of elementary models and conditions in a proper forcing.

Let us fix a large regular χ and consider elementary submodels of H_χ.

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- for such α, if M is a countable model in s and $\alpha \in M$ then $p(\alpha)$ is an M-generic condition for $F(\alpha)$.