On Woodin's HOD Conjecture, large cardinals beyond Choice, and class forcing

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The L-Dichotomy is resolved by large cardinals (e.g., the existence of a measurable cardinal) imply that the second alternative, in which L is far from V, is the true one.

Woodin's HOD Dichotomy theorem

Theorem (Woodin 2010[1])

If there exists an extendible cardinal, then either V is **close** *to HOD or is* **far** *from it.*

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1. *every singular cardinal* $\lambda > \kappa$ *is singular in* HOD *and* $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$, *or*
2. *every regular cardinal* $\lambda \geqslant \kappa$ *is* ω*-strongly measurable in* HOD.

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Moreover, the development of the **inner model program for a supercompact cardinal**, as carried out by Woodin, provides strong evidence for the first alternative of the Dichotomy.

The HOD Conjecture

Woodin's HOD Conjecture

The theory ZFC + "There exists an extendible cardinal" proves that there is a proper class of regular cardinals which are not ω-strongly measurable in HOD (hence the first alternative of the HOD Dichotomy holds, i.e., V is close to HOD).

Structural Reflection

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$SR(\mathcal{C})$: There exists a cardinal κ that reflects $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exist B in $\mathcal{C} \cap V_\kappa$ and an elementary embedding from B into A.

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Theorem

$SR(\Sigma_1)$ *holds, i.e.,* $SR(\mathcal{C})$ *holds for every* Σ_1 *definable class* $\mathcal{C}$.

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Theorem (Magidor 1970)

The following are equivalent:

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2. $\mathrm{SR}(\Sigma_3)$
3. *There exists an extendible cardinal.*

SR and the $\mathbb{L}$-Dichotomy

Let $\mathcal{C}$ be the Π_1 definable (without parameters) class of structures of the form $\langle L_\beta, \in, \gamma \rangle$, where γ and β are cardinals (in V) and $\gamma < \beta$.

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In the case of the HOD-Dichotomy the situation is completely different.

SR and the HOD-Dichotomy

Definition (Woodin 2010)

A transitive class model N of ZFC is a *weak extender model for the supercompactness of* κ if for every $\gamma > \kappa$ there exists a normal fine measure $\mathcal{U}$ on $\mathcal{P}_\kappa(\gamma)$ such that

1. $N \cap \mathcal{P}_\kappa(\gamma) \in \mathcal{U}$, and
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Theorem (Woodin 2010)

Suppose that κ *is an extendible cardinal. Then the following are equivalent.*

1. *The first alternative of the* HOD*-Dichotomy holds.*
2. *There is a weak extender model* N *for the supercompactness of* κ *such that* $N \subseteq \mathrm{HOD}$.
3. HOD *is a weak extender model for the supercompactness of* κ.

SR and the HOD-Dichotomy

In analogy with the L case, in which $SR(\mathcal{C})$, for a particular Π_1-definable class $\mathcal{C}$ of structures in L, yields the second alternative of the L-Dichotomy (i.e., L is far from V), one would expect, assuming the existence of an extendible cardinal, that $SR(\mathcal{C})$, for Π_1-definable clases $\mathcal{C}$ of structures in N, would fail strongly for any weak extender model N for a supercompact.

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But just the opposite holds:

Theorem

1. *If N is a weak extender model for δ supercompact, then $SR(\mathcal{C})$ holds for every Σ_2-definable class $\mathcal{C}$ of structures in N.*

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Theorem

1. *If N is a weak extender model for δ supercompact, then $SR(\mathcal{C})$ holds for every Σ_2-definable class $\mathcal{C}$ of structures in N.*
2. *If there exists a supercompact cardinal, then $SR(\mathcal{C})$ holds for every Σ_2-definable class $\mathcal{C}$ of structures in HOD.*

Transcendence over HOD

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Question

Is there any (natural) SR principle or, more generally, any large cardinal principle that would yield the second alternative to the HOD Dichotomy?

Large cardinals beyond Choice

Definition

A cardinal δ is a Berkeley cardinal if for every transitive set M such that $\delta \in M$ and every $\eta < \delta$ there exists an elementary embedding $j : M \to M$ with $\eta < \mathrm{crit}(j) < \delta$.

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Berkeley cardinals contradict the Axiom of Choice. Moreover, if δ_0 is the least Berkeley cardinal, then there exists $\gamma < \delta_0$ such that

$$V_\gamma \models \mathrm{ZF} + \text{“There exists a Reinhardt cardinal”}$$

The HOD Conjecture and Berkeley cardinals

Using some results from Woodin (2010) we showed the following:

Theorem (B.-Koellner-Woodin, 2018[3])

(ZF) If the HOD Conjecture holds, then there are no Berkeley cardinals.

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Theorem (B.-Koellner-Woodin, 2018[3])

(ZF) If the HOD Conjecture holds, then there are no Berkeley cardinals.

This points to a possible candidate for a large-cardinal principle compatible with ZFC that would yield the second alternative of the HOD Dichotomy.

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N-Berkeley cardinals

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What about when $N = \mathrm{HOD}$?

HOD-Berkeley cardinals

Theorem (Woodin)

Assume ZFC and that there exists an extendible cardinal. If there exists a HOD-Berkeley cardinal, then the second alternative of the HOD Dichotomy holds, hence HOD is far from V.

The HOD Conjecture and class forcing

Large cardinals are preserved by small forcing

Theorem (Levy-Solovay 1967)

All usual large cardinals are preserved by small (i.e., of size less than the cardinal) forcing notions. E.g., inaccessible, measurable, supercompact, etc.

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So, the general question is: What (big) forcing notions do preserve large cardinals?

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So, the general question is: What (big) forcing notions do preserve large cardinals?

For instance, does blowing up the power-set of κ preserve the large cardinal properties of κ?

Making a large cardinal indestructible

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Richard Laver (1978): If κ is a supercompact cardinal, then there is a forcing notion (the *Laver preparation*) that preserves the supercompactness of κ and makes it indestructible under further $< \kappa$-directed closed forcing.

Preserving Σ_3-correct cardinals

If κ is supercompact, then $V_\kappa \preceq_{\Sigma_2} V$. Hence, after the Laver preparation forcing,

$$V[G]_\kappa \preceq_{\Sigma_2} V[G]$$

for every V-generic filter $G \subseteq \mathbb{P}$, whenever $\mathbb{P}$ is $< \kappa$-directed closed.

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However, a similar Laver-indestructibility result for Σ_3-correct cardinals, and in particular for extendible cardinals, is not possible.

Theorem (B-Hamkins-Tsaprounis-Usuba 2015)

Suppose that $V_\kappa \prec_{\Sigma_2} V_\lambda$ *and* $G \subseteq \mathbb{P}$ *is a* V*-generic filter for nontrivial strategically* $<\kappa$*-closed forcing* $\mathbb{P} \in V_\eta$*, where* $\eta \leqslant \lambda$ *. Then for every* $\theta \geqslant \eta$*,*

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However, extendible cardinals, and even stronger large cardinal principles, implying Σ_n-correctness, $n \geqslant 3$, are preserved by suitable class-forcing iterations.

$C^{(n)}$-extendible cardinals

For each $n < \omega$, let $C^{(n)}$ be the Π_n-definable closed unbounded proper class of ordinals α that are Σ_n-correct, i.e., such that

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Definition

A cardinal κ is $C^{(n)}$-extendible (for $n \geqslant 1$) if for every $\lambda > \kappa$ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with critical point κ, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$.

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A cardinal κ is extendible iff it is $C^{(1)}$-extendible.

$C^{(n)}$-extendible cardinals and Vopěnka's Principle

Recall that Vopěnka's Principle (VP) is the schema asserting that for every (definable) proper class of structures of the same type there exist distinct A and B in the class with an elementary embedding $j : A \to B$.

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Theorem (B. 2012)

VP(Π_{n+1}), *namely* VP *restricted to classes of structures that are* Π_{n+1}*-definable, is equivalent to the existence of a* $C^{(n)}$*-extendible cardinal. Hence* VP *is equivalent to the existence of a* $C^{(n)}$*-extendible cardinal for each* $n \geqslant 1$.

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$\mathrm{VP}(\Pi_{n+1})$, *namely* VP *restricted to classes of structures that are* Π_{n+1}*-definable, is equivalent to the existence of a* $C^{(n)}$*-extendible cardinal. Hence* VP *is equivalent to the existence of a* $C^{(n)}$*-extendible cardinal for each* $n \geqslant 1$.

Brooke-Taylor (2011) shows that VP is indestructible under ORD-length iterations with Easton support of increasingly directed-closed forcing notions (*without the need of any preparatory forcing!*).

Preserving $C^{(n)}$-extendible cardinals under class forcing

Question

What ORD*-length forcing iterations preserve extendible and* $C^{(n)}$*-extendible cardinals?*

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The problem is how to lift (a proper class of) elementary embeddings of the form

$$j : V_\lambda \to V_\mu$$

witnessing the $C^{(n)}$-extendibility of $\mathrm{crit}(j)$, to

$$j : V_\lambda[G_\lambda] \to V_\mu[G_\mu]$$

where G is $\mathbb{P}$-generic over V.

Magidor's characterization of supercompact cardinals

Theorem (Magidor 1971)

For a cardinal δ, the following statements are equivalent:

1. *δ is a supercompact cardinal.*
2. *For every $\lambda > \delta$ in $C^{(1)}$ and for every $a \in V_\lambda$, there exist ordinals $\bar{\delta} < \bar{\lambda} < \delta$ and there exist some $\bar{a} \in V_{\bar{\lambda}}$ and an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:*
    - *$\mathrm{cp}(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.*
    - *$j(\bar{a}) = a$.*
    - *$\bar{\lambda} \in C^{(1)}$.*

Σ_n-supercompact cardinals

Definition

If $\lambda > \delta$ is in $C^{(n)}$, then we say that δ is λ-Σ_n-supercompact if for every $a \in V_\lambda$, there exist $\bar{\delta} < \bar{\lambda} < \delta$ and $\bar{a} \in V_{\bar{\lambda}}$, and there exists elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:

- $cp(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.
- $j(\bar{a}) = a$.
- $\bar{\lambda} \in C^{(n)}$.

We say that δ is Σ_n–supercompact if it is λ-Σ_n-supercompact for every $\lambda > \delta$ in $C^{(n)}$.

Theorem (Poveda 2018, Boney 2018)

A cardinal δ *is* Σ_{n+1}*-supercompact if and only if it is* $C^{(n)}$*-extendible.*

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In particular, a cardinal is extendible if and only if it is Σ_2-supercompact.

Lifting λ-Σ_n-supercompact embeddings

In a recent joint work with **A. Poveda** we make use of this characterization of $C^{(n)}$-extendibility to show that many ORD-length forcing iterations $\mathbb{P}$ preserve $C^{(n)}$-extendible cardinals.

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For this, one lifts ground model embeddings $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ witnessing the λ-Σ_{n+1}-supercompactness of δ to embeddings $j : V[G]_{\bar{\lambda}} \longrightarrow V[G]_\lambda$ verifying in $V[G]$ the same property.

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Key point: The cardinals λ for which this will be possible need to be sufficiently correct.

$\mathbb{P}$-reflecting cardinals

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A second reflection property of λ that will be required in our arguments is that

$$\langle V_\lambda, \in, \mathbb{P} \cap V_\lambda \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$$

for some big-enough k.

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for some big-enough k.

Let $C^{(k)}_{\mathbb{P}}$ be the closed and unbounded class of such cardinals λ.

A key lemma

The following is a key lemma:

Lemma

Suppose $\mathbb{P}$ *is a definable iteration. If* κ *is a* $\mathbb{P}$*-reflecting cardinal in* $C_{\mathbb{P}}^{(k)}$*, then* $\mathbb{P}$ *forces* $V[\dot{G}]_\kappa \prec_{\Sigma_k} V[\dot{G}]$.

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Lemma

Suppose $\mathbb{P}$ is a definable iteration. If κ is a $\mathbb{P}$-reflecting cardinal in $C_{\mathbb{P}}^{(k)}$, then $\mathbb{P}$ forces $V[\dot{G}]_\kappa \prec_{\Sigma_k} V[\dot{G}]$.

Thus, we give such cardinals a name:

Definition

A cardinal κ is $\mathbb{P}$-Σ_k-reflecting if it is $\mathbb{P}$-reflecting and belongs to $C_{\mathbb{P}}^{(k)}$.

$\mathbb{P}$-Σ_n-supercompactness

Definition (B.-Poveda 2018)

If $\mathbb{P}$ is a definable iteration, then we say that a cardinal δ is $\mathbb{P}$-Σ_n-supercompact if there exists a proper class of $\mathbb{P}$-Σ_n-reflecting cardinals, and for every such cardinal $\lambda > \delta$ and every $a \in V_\lambda$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and $\bar{a} \in V_{\bar{\lambda}}$, and there exists an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:

- $cp(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.
- $j(\bar{a}) = a$.
- $\bar{\lambda}$ is $\mathbb{P}$-Σ_n-reflecting.

Suitable iterations

Definition

A forcing iteration $\mathbb{P}$ is suitable if it is the direct limit of an Easton support iteration[4] $\langle \mathbb{P}_\lambda ; \dot{\mathbb{Q}}_\lambda : \lambda < \mathrm{ORD} \rangle$ such that for each λ,

1. If λ is an inaccessible cardinal, then $\mathbb{P}_\lambda \subseteq V_\lambda$.
2. There is some $\theta > \lambda$ such that

$$\Vdash_{\mathbb{P}_\nu} \text{“}\dot{\mathbb{Q}}_\nu \text{ is } \lambda\text{-directed closed”}$$

   for all $\nu \geqslant \theta$.

[4] Recall that an Easton support iteration is a forcing iteration where direct limits are taken at inaccessible stages and inverse limits elsewhere.

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Recall that a partial ordering $\mathbb{P}$ is weakly homogeneous if for any $p, q \in \mathbb{P}$ there is an automorphism π of $\mathbb{P}$ such that $\pi(p)$ and q are compatible.

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Main preservation theorem

Theorem (B.-Poveda 2018)

Suppose $m, n \geqslant 1$ and $m \leqslant n+1$. Suppose $\mathbb{P}$ is a weakly homogeneous Γ_m-definable suitable iteration and there exists a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals. If δ is a $\mathbb{P}$-Σ_{n+1}-supercompact cardinal, then

$$\Vdash_{\mathbb{P}} \text{“}\,\delta \textit{ is } C^{(n)}\textit{-extendible”}.$$

Preserving VP level-by-level

Theorem (Brooke-Taylor 2011)

Let $\mathbb{P}$ be a definable suitable iteration. If VP holds in V, then VP holds in $V^{\mathbb{P}}$.

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Theorem

Let $n, m \geqslant 1$ be such that $m \leqslant n+1$, and let $\mathbb{P}$ be a weakly thomogeneous Γ_m-definable suitable iteration. Then,

1. *If $\Gamma = \Sigma$ or $n > 1$, and $\mathsf{VP}(\Pi_{m+n})$ holds, then $\mathsf{VP}(\Pi_{n+1})$ holds in $V^{\mathbb{P}}$.*
2. *If $\Gamma = \Pi$ and $n = 1$, $\mathsf{VP}(\Pi_{m+1})$ holds, and ORD is Π_{m+2}-Mahlo, then $\mathsf{VP}(\Pi_2)$ holds in $V^{\mathbb{P}}$.*

$C^{(n)}$-extendible cardinals and the GCH

Let $\mathbb{P} = \langle \mathbb{P}_\alpha ; \dot{\mathbb{Q}}_\alpha : \alpha \in \mathrm{ORD} \rangle$ be the standard Jensen's proper class iteration for forcing the global GCH. Namely, the direct limit of the iteration with Easton support where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\Vdash_{\mathbb{P}_\alpha}$ "α is an uncountable cardinal", then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \mathrm{Add}(\alpha^+, 1)$", and $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial", otherwise.

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$\mathbb{P}$ is weakly homogeneous, suitable, and Π_1-definable.

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Theorem (Tsaprounis 2013)

Forcing with $\mathbb{P}$ preserves $C^{(n)}$-extendible cardinals.

Changing the power-set function on regular cardinals

A class function E from the class REG of infinite regular cardinals to the class of cardinals is an Easton function if:

1. $\mathrm{cf}(E(\kappa)) > \kappa$, for all $\kappa \in \mathrm{REG}$
2. If $\kappa \leqslant \lambda$, then $F(\kappa) \leqslant F(\lambda)$

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If the GCH holds in the ground model, then $\mathbb{P}_E$ preserves all cardinals and cofinalities and forces that $2^\kappa = E(\kappa)$ for every regular cardinal κ.

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$\mathbb{P}_E$ is suitable and weakly homogeneous.

Changing the power-set function on regular cardinals

Theorem (B.-Poveda 2018)

If E is a Δ_2-definable Easton function, then $\mathbb{P}_E$ preserves $C^{(n)}$-extendible cardinals, all $n \geqslant 1$. More generally, if E is a Π_m-definable Easton function ($m > 1$) and λ is $C^{(m+n-1)}$-extendible, then $\mathbb{P}_E$ forces that λ is $C^{(n)}$-extendible, all $n \geqslant 1$ such that $m \leqslant n+1$.

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The theorem is sharp: If κ is the least $C^{(n)}$-extendible cardinal, then the Easton function E that sends $\aleph_0$ to κ and every uncountable regular cardinal λ to $\max\{\lambda^+, \kappa\}$ is Π_{n+2}-definable and destroys κ being inaccessible. In the case $n = 1$ this gives in fact an example of a Π_2-definable Easton function E such that $\mathbb{P}_E$ destroys an extendible cardinal.

Forcing V "far" from HOD

Let $\mathbb{C} = \langle \mathbb{P}_\alpha ; \dot{\mathbb{Q}}_\alpha : \alpha \in \mathrm{ORD} \rangle$ be the Easton support iteration where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\Vdash_{\mathbb{P}_\alpha}$ "α is regular" then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{Coll}}(\alpha, \alpha^+)$", and $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial" otherwise.

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Forcing with $\mathbb{C}$ preserves $C^{(n)}$-extendible cardinals (hence also VP) and forces $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$, for every regular cardinal λ.

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Note: Forcing $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$, for some singular cardinal λ, while preserving some extendible cardinal smaller than λ would refute Woodin's HOD Conjecture.

Forcing further disagreement between V and HOD

Let K be a function on the class of infinite cardinals such that $K(\lambda) > \lambda$, for every λ, and K is increasingly monotone. Let $\mathbb{P}_K$ be the direct limit of an iteration $\langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha : \alpha \in \mathrm{ORD} \rangle$ with Easton support where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\Vdash_{\mathbb{P}_\alpha}$ "α is regular" then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \mathrm{Coll}(\alpha, K(\alpha))$", and $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial" otherwise.

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$\mathbb{P}_K$ preserves all inaccessible cardinals that are closed under K. Moreover, for each α such that $\Vdash_{\mathbb{P}_\alpha}$ "α is regular", the remaining part of the iteration after stage α is α-closed, hence it preserves α. Also, if K is Π_m-definable ($m \geqslant 1$), then $\mathbb{P}_K$ is also Π_m-definable.

Theorem (B.-Poveda 2018)

If K *is* Δ_2*-definable, then* $\mathbb{P}_K$ *preserves* $C^{(n)}$*-extendible cardinals, all* $n \geqslant 1$*. More generally, if* K *is* Π_m*-definable* ($m > 1$) *and* λ *is* $C^{(m+n-1)}$*-extendible, then* $\mathbb{P}_K$ *forces that* λ *is* $C^{(n)}$*-extendible, all* $n \geqslant 1$ *such that* $m \leqslant n+1$*. Moreover,* $\mathbb{P}_K$ *forces*

$$(\lambda^+)^{\mathsf{HOD}} \leqslant K(\lambda) < \lambda^+$$

for all infinite regular cardinals λ*.*

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For example, let K be such that $K(\lambda)$ is the least singular cardinal in HOD greater than λ, i.e., $K(\lambda) = (\lambda^{+\omega})^{\mathsf{HOD}}$. Then, K is Δ_2-definable, and we have the following.

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Corollary

$\mathbb{P}_K$ *preserves extendible cardinals and forces*

$$(\lambda^{+\omega})^{\mathsf{HOD}} < \lambda^+$$

for every regular cardinal λ.