Hilbert's 16th problem and o-minimality

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Are there other qualitative phenomena needed to describe the phase portrait of F?

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- **Dulac's problem:** if F extends to a real analytic vector field on the sphere S^2, then F has finitely many limit cycles.
- **Hilbert's 16th problem (H16):** if F is polynomial of degree d, there exists $H(d) \in \mathbb{N}$, depending only on d, such that F has at most $H(d)$ many limit cycles.

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1963

Ilyashenko and Novikov produce the first counterexamples to Petrovskii and Landis's solution.

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Ecalle and Ilyashenko independently publish papers that fill the gap in Dulac's proof. Both of these gap-filling proofs are much longer than Dulac's original proof, but show that Dulac's original argument was right, "just" incomplete.

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The problem is: the Poincaré map $r(x)$ is not even solution of any reasonably simple differential equation.
In the case of Poincaré's example, the first return map is real analytic at 0, so there are only finitely many limit cycles near the cycle C.

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While Dulac completed Point 1, Point 2 was the gap left unproved by him and proved 70 years later by Ecalle and Ilyashenko.

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For each degree d, let $\mathcal{S}_d$ be the collection of all polynomial vector fields in the plane of degree d (a definable family in $\overline{\mathbb{R}}$).

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Good news: Roussarie shows that if all singularities of F_μ are isolated, then P is always a polycycle.

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Fact

There are open neighbourhoods U of μ and V of P such that the transition maps $f_{\mu',i}$ and $g_{\mu',i}$ are well defined for all parameters $\mu' \in U$ and segment coordinates $x_i, y_i \in V$.

definability

So $x_1 \in I_1 \cap V$ corresponds to a limit cycle of $F_{\mu'}$ near P, with $\mu' \in U$, if and only if x belongs to the set $A^P_{\mu'}$ of all isolated points of the set

$$B^P_{\mu'} := \{x_1 \in I_1 \cap V : \text{ there exist } x_2, \ldots, x_{k+1}, y_1, \ldots, y_k \text{ such that } y_i = g_{\mu',i}(x_i) \text{ and } x_{i+1} = f_{\mu',i}(y_i) \text{ for each } i, \text{ and } x_{k+1} = x_1\},$$

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So: Let $\mathbb{R}_{\text{trans}}$ be the expansion of the real field by the parametric transition maps associated to every limit periodic set of every F_μ in $\mathcal{S}_d$ as above.

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So FCC follows from (UF) and the following:

Conjecture (o-minimality)

The structure $\mathbb{R}_{\text{trans}}$ is o-minimal.

Let $\mathcal{NRH}_d$ be the subfamily of all vector fields in $\mathcal{S}_d$ that have only *non-resonant hyperbolic* singularities. Let $\mathbb{R}_{\text{nrh}}$ be the expansion of $\overline{\mathbb{R}}$ generated by all parametric transition maps associated to the vector fields in $\mathcal{NRH}_d$, for $d \in \mathbb{N}$.

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