Kripke Semantics, C and BL

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January 23, 2019

Abstract

We investigate intermediate logics having a weak form of contraction. Whereas intermediate logics are generally constructive and well-understood proof-theoretically, the same cannot be said for logics with restricted contraction, having a semantic motivation; as such, these logics are generally classed as 'fuzzy.' Generalized Basic Logic (GBL) is one such logic, restricting the Basic Logic (BL) of Hajek by omitting pre-linearity from the axioms. We have succeeded in extending an algebraic semantics of Urquhart to BL (Hajek's Basic Logic), have proven adequacy for BL under this semantics.

What makes a logic Substructural?

- Resultant from removing one or more of the structural rules – contraction (Affine logics), weakening (Relevance logics), contraction and weakening (Linear Logic), Contraction and weakening and commutativity (Lambek Calculus); Contraction and commutativity (Minimal Logic)
- Restrictions: Just one formula on the right of the turnstyle (Intuitionistic logic, Intuitionistic Linear Logic); restricted contraction (Lukasiewicz logic, Intermediate logics)
- Sometimes the motivation is purely algebraic or semantic, and then one "finds" the proof theory: e.g. Gaggles (Dunn), commuting equivalence relations (Rota), Quantales . . .
- And sometimes directly from the combinators: BCK logic

Connections between Fuzziness and Intuitionism

- Affine logics usually reject contraction and have a fundamentally different proof theory than other non-classical logics
- Reject excluded middle, double negation equivalences, etc.
- *Sorites Paradox* (example); cannot be expressed in classical systems because of semantics
- *Sorites* can be expressed, but is not derivable in e.g. Lukasiewicz logic
- This means deduction theorem fails for these logics.
- Hence the usual analytic proof systems are out in the fuzzy case.

Connections Continued

- But Intuitionists also reject classical rules!
- They reject excluded middle, double negation elimination, etc.
- But they also reject the classical version of cut – they put constructive conditions on choosing witnesses or parameters of a function for instance; and the constructive conditional is generally not identical to that of the classical conditional anyway
- And the whole point of cut-elimination for **LJ** is showing that cuts can be eliminated entirely in favour fully explicit, finitary derivations that only use constructive principles

Intuitionistic Logic – LJ

- $\phi \to \phi$ – Identity
- $(\phi \to \psi) \to (\psi \to \chi) \to \phi \to \chi$ – Transitivity
- $\phi \wedge \psi \to \phi$ and $\phi \wedge \psi \to \psi$ – $\wedge$-elimination
- $\phi \wedge \psi \to \psi \wedge \phi$ – commutativity of $\wedge$
- $\phi \to \phi \vee \psi$ and $\psi \to \phi \vee \psi$ – $\vee$-intro
- $(\phi \to \psi) \wedge (\chi \to \psi) \to ((\phi \vee \chi) \to \psi)$ – $\vee$-elimination
- $\perp \to \phi$ – Ex Falso Quodlibet
- Rules: Substitution and Modus Ponens: From $\vdash \phi$ and $\vdash \phi \to \psi$ then $\vdash \psi$

Intuitionistic Logic and Intermediate Logics

Point 1

Add Excluded Middle $A \vee \neg A$ and you have classical logic. Intuitionistic Logic therefore proves no classical tautologies.

Point 2

If you add Peirce's law, Linearity, weakened excluded middle etc. you get an intermediate logic – a logic weaker than classical logic, but stronger than Intuitionist logic (because you have at least one more classical tautology now but not the full classical system).

Point 3

It turns out there are $\mathfrak{c}$-many such non-conservative extensions of Intuitionist logic (Jankov, 1968); curiously, they form a complete lattice, with classical logic at the top – the least upper bound of the process of adding classical principles – and intuitionist logic at the bottom.

Intuitionistic Logic and Semantics

Point 4

Intuitionistic logic, unlike classical logic, has a variety of options for the semantics – blessing and curse.

Point 5

Heyting Algebras; Kripke semantics; Beth semantics; Computations (simply-typed lambda calculus). . . "negative translations" which effectively show that classical logic and constructive logic coincide in negative contexts . . .

Point 6

. . . And Matrices? Wait, is **LJ** many-valued? And what about Intermediate logics?

Point 7

The most popular alternative (even now) is *Forcing*, based on Kripke Frames. It is probably easier to motivate than topological semantics, Heyting Algebra, or anything else.

Point 8

The idea is interesting, and ultimately is inspired in some way by the original views of Brouwer in which mathematics is a creative subject on the part of a mathematician whose knowledge increases with time.

Kripke frames and forcing continued

Point 10

The essential defining features of forcing in the Kripke structures are: (1) Monotonicity of valuations and (2) "Eternity condition" i.e. once a formula is valued true, it's *always* true.

Point 11

Intuitively, the mathematician must be consistent in his evaluations and once he *knows* he has a proof that is valid (or at least type-checks!), that proof holds *eternally*.

point 9

Granted, Forcing in Kripke structures is fundamentally ambiguous about what counts as *constructive* proof, and locally behaves *classically* (counterintuitive).

Definition

A Kripke frame is $P = \langle X, \leq \rangle$, with P a possibly empty set, and $\leq$ a binary relation on P. Elements of P are nodes, and $\leq$ is known as accessibility relation on P.

Definition

A Kripke semantics, $P = \langle X, \leq, \Vdash \rangle$, consists of a Kripke frame $P = \langle X, \leq \rangle$ and $\Vdash$ is a relation on nodes satisfying the following conditions:

- w$\Vdash$ A $\wedge$ B iff w$\Vdash$ A and w$\Vdash$B
- w$\Vdash$ A $\vee$ B iff w$\Vdash$ A or w$\Vdash$B
- never w$\Vdash\perp$
- w$\Vdash$ $A \rightarrow B$ iff $\forall w\prime$ such that $w \leq w\prime : w\prime \Vdash A$ then $w\prime \vdash B$
- w$\Vdash$ $\neg$A if $\forall w\prime : w \leq w\prime, w\prime \nVdash$ A (after setting $\neg A = A \rightarrow \perp$)

Intuitionistic Logic and Semantics: Heyting Algebra

- Distributive lattice with respect to $\top, \bot, \vee, \wedge$
- $a \wedge (a \supset b) = a \wedge b$
- $(a \supset b) \wedge b = b$
- $(a \supset b) \wedge (a \supset c) = a \supset (b \wedge c)$
- $\bot \wedge a = \bot$
- $\bot \supset \bot = \top$
- Complement defined as follows: $a\prime$ is defined as $a\prime = a \supset \bot$

Intuitionistic Logic and Semantics: Proof that [0,1] is a Heyting Algebra

Proof

Take the x, y from the unit [0,1]. Then the following definitions yield a Heyting Algebra:

- $x \wedge y = Min(x, y)$
- $x \vee y = Max(x, y)$
- $x \supset y = 1$ if $x \leq y$ and y otherwise.
- $\top = 1$ and $\bot = 0$.

Godel's First Theorem, ca. 1932

Theorem (LJ is not finitely many-valued.)

Let **LJ** *be as given above, and let* $Th(\mathbf{LJ})$ *be the set of all formulas provable from* **LJ**. *There is no finite model* $\mathfrak{M}$ *for which* $Th(\mathbf{LJ})$, *and only formulas in* $Th(\mathbf{LJ})$, *are satisfied (that is, yield designated values for an arbitrary assignment).*

Proof

Assume **LJ** is an n-valued logic, i.e. has only finite models. Since $A \leftrightarrow A$ is **LJ**-valid, if A and B have the same truth value, $A \leftrightarrow B$ must have the same truth value. Since there are only n values, the following sentence constructed out of $n+1$ atoms is valid: $(p_1 \leftrightarrow p_2) \vee \ldots \vee (p_1 \leftrightarrow p_n) \vee (p_2 \leftrightarrow p_3) \vee \ldots \vee (p_n \leftrightarrow p_n{+}_1)$ (It says that at least two of the atoms share their truth value.) Since there are $n+1$ atoms, this must be so under any assignment of values to atoms, since there are only n values. But since **LJ** has *the disjunction property*, it follows that one of the disjuncts is valid; say $p_i \leftrightarrow p_j$. Since $i \neq j$ (given the construction of the disjunction), there is an assignment giving p_i and p_j different values, making $p_i \leftrightarrow p_j$ false. Contradiction.

Intuitionistic Logic and Intermediate Logics Continued

Point 7

In showing that **LJ** wasn't characterizable by a finite matrix, together with the fact that **LJ** is consistent (and so has a model), tells us that it is characterized by infinite matrices. (This confirmed a conjecture of Oskar Becker in 1927 that **LJ** is a many-valued logic.)

Point 8

Godel's proof tacitly appeals to excluded middle. So a thoroughgoing constructivist might reject this and other non-constructive proofs in the metatheory. I'm not sure this is something Godel cared about, but some logicians do (e.g. Consider the debate between Burgess and the Relevantists in the 1980's).

Intuitionistic Logic and Intermediate Logics Continued

Point 10

To this end, Jaskoski specifically constructed an infinite-valued characteristic matrix for **LJ** in 1936. The "truth-degrees," however, do not have a simple explanation.

Point 11

So **LJ** has a fuzzy reading.

Point 12

One sees a patent duality between mainstream fuzzy logics and **LJ**: the former has a very well-understood variety of semantics, and is motivated semantically, and the latter has a variety of semantic approaches, each of which have quirks, with a very well-understood proof-theoretic motivation.

Godel's Second Theorem

Theorem (LJ has infinitely many non-conservative extensions.)

Infinitely many systems lie between **LJ** *and Classical logic, all of which include LJ as a subset and are included in Classical logic as subsets.*

Intuitionistic Logic and Intermediate Logics

Note 1

One of the simplest such extensions, formed by adding Linearity to **LJ**: $(A \to B) \vee (B \to A)$ or **GD**, is known as Godel-Dummett Logic. Dummett (1959) proved this logic is complete. It can also be proven complete for linearly-ordered Kripke models.

Note 2

If you take as a natural model of **LJ** the unit [0,1] with the underlying structure being a Heyting Algebra (i.e. 1 is designated and everything else undesignated), then what we have when linearity is added is a chain structure for which the resulting logic **GD** is complete, coinciding with standard structural reading of the closed unit interval.

Note 3

But this formula is unprovable in **LJ**, and so **LJ** is not complete with respect to [0,1] with added chain condition.

Our work

Goals

We aim to generalize Kripke structures in a way that fits **GBL**; extend Urquhart's logic **C** to more well-known **BL**, showing such an extension with intended interpretation is complete and sound; and we'd like to find a way to relate **BM** frames to Totally-Ordered Commutative Monoids (*TOCOM*).

Fuzzy Logics

Note 1

Often called "logics of the unit-interval" [0,1], and fuzzy logics since (taking after Zadeh) have conformed to this paradigm.

Note 2

BUT they needn't be: one can take countable sub-intervals like the rationals. Indeed, this was Lukasiewicz's first infinite-valued logic, $L_{\aleph}$.

Note 3

Our focus here is the most important generalization of several fuzzy logics, Hajek's **BL**.

GBL

(A1) $\phi \to \phi$ (*identity*)

(A2) $(\phi \to \psi) \to (\psi \to \chi) \to \phi \to \chi$ (*composition*)

(A3) $\phi \otimes \psi \to \psi \otimes \phi$ (*commutativity of strong conjunction*)

(A4) $\phi \otimes \psi \to \psi$ (*projection*)

(A5) $(\phi \otimes \psi \to \chi) \leftrightarrow (\phi \to \psi \to \chi)$ (*currying* and *uncurrying*)

(A6) $\phi \wedge \psi \leftrightarrow \phi \otimes (\phi \to \psi)$ (*weak conjunction*)

(A7) $\phi \wedge \psi \to \psi \wedge \phi$ (*commutativity of weak conjunction*)

(A8) $\phi \to \phi \vee \psi$ and $\psi \to \phi \vee \psi$ (*disjunction introduction*)

(A9) $(\phi \to \psi) \wedge (\chi \to \psi) \to \phi \vee \chi \to \psi$ (*disjunction elimination*)

(A10) $\perp \to \phi$ (*efq*)

GBL and BL

Note 1

When you add to **GBL** linearity, we get **BL**.

Note 2

This means that **BL** is both mildly constructive and fuzzy.

Note 3

And **GBL** is constructive and fuzzy.

Note 4

Well, if your logic is intuitionistic, you can give a Kripke semantics for it via forcing. But what does that look like in the fuzzy case?

MV chains

Definition

Let $\mathcal{L}$ be the language of **GBL** algebras, i.e. the terms constructed via $\top, \bot, x \otimes y, x \wedge y, x \vee y$, and $x \to y$.

MV-chain

The standard **MV**-chain, denoted $[0,1]_{MV}$, is the **MV**-algebra defined as

- The domain of $[0,1]_{MV}$ is the unit interval $[0,1]$
- $\top = 1$
- $\bot = 0$
- $x \otimes y = \max\{0, x + y - 1\}$
- $x \wedge y = \min\{x, y\}$
- $x \vee y = \max\{x, y\}$
- $x \to y = \min\{1 - x + y, 0\}$

Bova-Montagna and **GBL**

Note 5

Introduce Bova-Montagna's paper on poset-sums and **GBL** algebras.

Note 6

In Bova-Montagna's paper (2009), the authors show consequence in commutative, bounded quasiequational GBL is decidable and in *P-SPACE* (in contrast to the noncommutative case which is undecidable, and the quasiequational theory in the variety of GBL which is also undecidable), and they give an exponential bound on computing countermodels on terms in the algebra.

Kripke Semantics for GBL

note 7

The definition they give for Poset sums there is *very* similar to that of forcing for Kripke structures. A little too similar. . .

note 8

We use the term *frame* since, these seem to generalise Kripke frames. And indeed, that's what we've shown!

Kripke Semantics of *GBL*

Given a BM-frame $BM = \langle W, \geq, \Vdash \rangle$ the valuation function $\Vdash$ can be extended to all formulas as:

$$
\begin{aligned}
w \Vdash \top &= 1 \\
w \Vdash \bot &= 0 \\
w \Vdash \phi \otimes \psi &= (w \Vdash \phi) \otimes (w \Vdash \psi) \\
w \Vdash \phi \wedge \psi &= (w \Vdash \phi) \wedge (w \Vdash \psi) \\
w \Vdash \phi \vee \psi &= (w \Vdash \phi) \vee (w \Vdash \psi) \\
w \Vdash \phi \to \psi &= \sup_{w' \geq w} ((w' \Vdash \phi) \to (w' \Vdash \psi))
\end{aligned}
$$

where the operations on the right-hand side are the operations on the standard MV-chain $[0,1]_{MV}$.

Kripke Semantics for GBL

Definition

A BM frame is a triple $BM = \langle W, \geq, \Vdash \rangle$ where $(W, \geq)$ is a poset, and $\Vdash$ is a mapping of type $W \to At \to [0,1]_{MV}$ satisfying:
(i) If $v \geq w$ then $w \Vdash p \geq v \Vdash p$
(ii) If $\exists v : v \Vdash p \in (0,1)$ then $\forall w < v((w \Vdash p) = 0)$ and $\forall w > v((w \Vdash p) = 1)$ where $p \in At$ are atomic formulas.

Kripke Semantics for GBL

Theorem

Kripke semantics is the particular case of the BM semantics when the BM frames are restricted to Kripke frames.

Proof:

Any Kripke frame is also a Bova-Montagna frame: Kripke frames are the particular case when the valuations $w \Vdash p \in [0,1]$ are always in the finite set $\{0,1\}$. These can then be identified with the Booleans.

Kripke Semantics for **GBL**

Theorem

$\vdash \phi_{GBL}$ iff for all BM frames we have that $w \Vdash \phi = 1$, $\forall w \in W$.

Urquhart Semantics and **C**

Note 1

The other direction of our research is involves extending Urquhart's logic **C**, a many-valued logic whose semantics is algebraic.

Note 2

There are many interesting things about this system and what he saw in it. For instance, Urquhart has certain philosophical objections to the standard many-valued logics. And in the feature of linearity, **C** shares a lot with relevant systems (R-mingle for example).

Note 3

For one, the semantics are pretty general – build a totally ordered commutative monoid that is additive – and this is decades before researchers started using these things (or lattice-ordered groupoids, etc. for that matter)!

Urquhart Semantics and **C**

Note 4

Fundamentally, his **C** is kind of a meta-fuzzy logic. It's non-numeric, although different structures could easily fit inside it. In some way, we could probably think of this as a kind of attempt at **BL**, pre-Hajek's **BL**.

Note 5

The underlying algebra is additive, and has many of the more general features of Chang's **MV** algebra. It is also a constructive logic, with linearity added; and so a fragment of the algebra coincides with Heyting Algebras.

Note 6

Noticeably, there is a lack of additive/multiplicative residuation unlike most fuzzy systems.

Urquhart Semantics and **C**

Note 7

He proves **C** (presented next page) complete and sound with respect to his algebraic semantics.

Note 8

We noticed that adding tensor product axioms yields **BL**. So we have extended his soundness and completeness theorems to **BL**, with some minor additions.

Note 9

Short of improving their upper bound, we think a purely logical proof of decidability would be nice. A logical proof would be a simpler proof and would fit nicely into the literature of hypersequents . . .

Urquhart's **C**

Definitions

Urquhart gives the following Hilbert-style presentation of his system (with substitution and modus ponens):

(1)$\phi \to (\psi \to \phi)$

(2) $(\phi \to \psi) \to ((\theta \to \phi) \to (\theta \to \psi))$

(3)$\phi \to (\theta \to \psi) \to \theta \to (\phi \to \psi)$

(4) $(\phi \wedge \psi) \to \phi$

(5) $(\phi \wedge \psi) \to \psi$

(6) $\phi \to (\psi \to (\phi \wedge \psi))$

(7) $\phi \to (\phi \vee \psi)$

(8) $\psi \to (\phi \vee \psi)$

(9)$((\phi \to \psi) \wedge (\theta \to \psi)) \to (\phi \vee \theta) \to \psi)$

(10) $(\phi \to \psi) \vee (\psi \to \phi)$

TOCOM's for **C**

Definition

TOCOM $A = \langle A, +, 0, \leq, \rangle$, or a totally ordered commutative monoid, is an algebra on A with an associative, commutative operation $+$, and a neutral element 0 i.e. $x + 0 = x$ for all $x \in A$. Additionally, $\leq$ is a total ordering on A, such that it is monotone with respect to addition in the structure i.e. if $x \leq y$ then $x + z \leq y + z$ for any $z \in A$.

TOCOM's for C

Definitions

[Urquhart semantics] A model $\mathfrak{M}$ over A consists of [P] $\subset$ A for every propositional variable P. These sets [P] are required to be increasing in the following sense: for $B \subset A$, and $x \in B$, if x≤y then y∈B. Define inductively (i.e. truth at a point $x \in A$):

- $x \models P$ iff $x \in [P]$
- $x \models \phi \wedge \psi$ iff $x \models \phi$ and $x \models \psi$
- $x \models \phi \vee \psi$ iff x $\models \phi$ or ψ
- $x \models \phi \rightarrow \psi$ iff ∀y ∈ A such that y $\models \phi$, then x + y $\models \psi$

Fact

The set of points at which a formula ϕ is true are upwards closed sets.

Urquhart's Theorem

Theorem (C Adequacy)

C is sound and complete for TOCOM's.

C can be extended to BL

Theorem (C with tensor is BL)

C, extended with tensor-product axioms, is BL.

Extending C to BL

C extended with tensor

The following axioms, when added to C, yields BL:

(A3) $\phi \otimes \psi \to \psi \otimes \phi$ (*commutativity of strong conjunction*)

(A4) $\phi \otimes \psi \to \psi$ (*projection*)

(A5) $(\phi \otimes \psi \to \chi) \leftrightarrow (\phi \to \psi \to \chi)$ (*currying* and *uncurrying*)

(A6) $\phi \wedge \psi \leftrightarrow \phi \otimes (\phi \to \psi)$ (*weak conjunction*)

Extending C to BL

Problem

The natural next step is to show BL is complete with respect to TOCOM's. The problem is that TOCOM's don't have residuation, and Hajek's book pretty clearly demonstrates that BL's underlying algebra is a residuated lattice that is linearly ordered.

Solution

So we are halfway there. We need to add (i) divisibility to TOCOM's, and we also need to add (ii) $\wedge$-completeness:

- i If $x \geq y$ then there exists a z such that $x = y + z$ and
- ii Let Z be the set of z's satisfying divisibility; then every X such that X $\subset$ Z has an infimum, denoted $\wedge X$.

Adequacy for *BL*

Theorem (*BL* is adequate for TOCOMS)

BL is sound and complete for TOCOM's with divisibility and $\wedge$-completeness.

The End