TUTORIAL ON FORCING

PART 1: BASICS OF FORCING AND WHAT ONE CAN DO EVEN WITH A SIMPLE FORCING

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Theme of the day:

forcing is an induction over a partial order

SET THEORY IS THE SCIENCE OF DOING INDUCTION, somewhat like the classical recursion theory.

WELL ORDER ≡ AN ORDER ALONG WHICH INDUCTIVE CONSTRUCTIONS WORK ≡ EVERY NON-EMPTY SUBSET HAS THE LEAST ELEMENT, SO WE KNOW AT ALL TIMES WHAT TO DO NEXT

Let us start with an example of transfinite induction, which illustrates this point of view.

Th (Mazurkiewicz 1930s)
There is $A \subseteq \mathbb{R}^2$ such that for every line ℓ in $\mathbb{R}^2$, $|A \cap \ell| = 2$.

Proof We list our

- REQUIREMENTS: all lines, a set of size $\mathfrak{c}$
  $\{ \ell_\alpha : \alpha < \mathfrak{c} \}$, $|A \cap \ell_\alpha| = 2$

We construct our set by

- APPROXIMATIONS
  $\{ A_\alpha : \alpha < \mathfrak{c} \}$, $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$, $A = \bigcup_{\alpha < \mathfrak{c}} A_\alpha$

by the method of

- TRANSFINITE INDUCTION
  At the stage α, let $X_\alpha = \bigcup \{ \ell_\gamma \cap \ell_\alpha : \gamma < \alpha \}$,
  so $|X_\alpha| < \mathfrak{c}$ (we assume $\gamma \neq \beta \Rightarrow \ell_\gamma \neq \ell_\beta$).
  - INDUCTIVE HYPOTHESES: no 3 points in $A^0_\alpha = \bigcup_{\gamma < \alpha} A_\gamma$ are colinear and $|A^0_\alpha| < \mathfrak{c}$

Let $S_\alpha = \{ \ell : \ell$ a line determined by $X_\alpha \}$

Now consider $R_\alpha = \ell_\alpha \setminus (\bigcup S_\alpha \cup A^0_\alpha)$, it is of size $\mathfrak{c}$, and choose 0, 1 or 2 points from R_α to complete A^0_α to A_α. ∎ QED

INDUCTION ALONG $\mathfrak{c}$

TOWARDS COHEN FORCING

When we say a "real" we actually may mean an element of ${}^{\omega}\omega$ or ${}^{\omega}2$, using standard identifications. Here we shall work with

$${}^{\omega}2 = \{ f : f \text{ a function from } \omega \text{ to } 2 \}$$

COMBINATORIAL PROBLEM

Given a family $\mathcal{F} \subseteq {}^{\omega}2$, suppose we want to construct $f \in {}^{\omega}2 \setminus \mathcal{F}$.

If $\mathcal{F}$ is countable, we could view this as an induction

$$\mathcal{F} = \{ f_n : n < \omega \}$$

REQUIREMENTS $\quad f \neq f_n \quad \forall n$

and then construct f by inductively choosing $f(n) \neq f_n(n)$

"DIAGONALISATION".

If $\mathcal{F}$ is not known to be countable, this is not possible, the DIAGONAL is too SHORT.

So let us try a different way...

INDUCTION OVER A PARTIAL ORDER

We are going to replace an increasing ω-sequence of approximations by

- APPROXIMATIONS ALONG A PARTIAL ORDER

$$\mathbb{P} = \{ p : \omega \to 2 \mid p \text{ a partial function, } \operatorname{dom}(p) \text{ finite} \}$$

ordered by $p \leq q$ iff $p \subseteq q$

- REQUIREMENTS

$f \neq g$ for all $g \in \mathcal{F}$

DEF 1

A FILTER IN $\mathbb{P}$ is a set of coherent elements of $\mathbb{P}$ ≡ CONDITIONS

i.e. $H \subseteq \mathbb{P}$ is a filter if

a) $(\forall p, q \in H)(\exists r \in H)\; p, q \leq r$

b) $p \leq q,\ q \in H \Rightarrow p \in H$

c) $H \neq \emptyset$.

Lemma If $H \subseteq \mathbb{P}$ is a filter, then $\bigcup H$ is a partial function from ω to 2.

Proof If $n \in \operatorname{dom}(\bigcup H)$, then $\bigcup H(n)$ is uniquely determined by (b) in Def 1.

To meet the requirements, we use the idea of DENSE SETS.

Def 2

A subset $\mathcal{D} \subseteq \mathbb{P}$ is DENSE if $(\forall p \in \mathbb{P})(\exists q \in \mathcal{D})\ p \leq q$ ∎

} COFINAL

We can express the requirements using dense sets

- $\mathcal{D}_g = \{ p \in \mathbb{P} : \exists n \in dom(p)\ \ p(n) \neq g(n) \}$ for $g \in \mathcal{F}$

  note that each $\mathcal{D}_g$ is dense as if $p \in \mathbb{P}$, it suffices to find $n \notin dom(p)$ and define $q = p \cup \{ (n, 1 \dot{-} g(n)) \}$

- $\mathcal{E}_n = \{ p \in \mathbb{P} : n \in dom(p) \}$ for $n \in \omega$ also dense

Lemma 3 If $H \subseteq \mathbb{P}$ is a filter such that $H \cap \mathcal{D}_g \neq \emptyset$ for all $g \in \mathcal{F}$ and $H \cap \mathcal{E}_n \neq \emptyset$ for all $n \in \omega$, then $\bigcup H \in {}^{\omega}2 \setminus \mathcal{F}$. ∎

Lemma 4

If $\mathcal{F}$ is countable, then there is a filter H such that $H \cap \mathcal{D}_g \neq \emptyset$ for all $g \in \mathcal{F}$ and $H \cap \mathcal{E}_n \neq \emptyset$ for all $n \in \omega$.

Proof DIAGONALISATION

Let $\mathcal{F} = \{g_n : n < \omega\}$. By induction on $n < \omega$ construct

$$p_0 \leq p_1 \leq \dots$$

such that $p_{n+1} \in \mathcal{D}_{g_n} \cap \mathcal{E}_n$.

- $p_0 = \emptyset$
- Given p_n, let $p_n' \in \mathcal{D}_{g_n}$ with $p_n' \geq p_n$ (exists as $\mathcal{D}_{g_n}$ is dense). Then let $p_{n+1} \geq p_n'$ with $p_{n+1} \in \mathcal{E}_n$, using the density of $\mathcal{E}_n$.

Now let $H = \{ q \in \mathbb{P} : (\exists n < \omega)\ q \leq p_n \}$.

Exercise H is a filter.

ADDING A COHEN REAL/OR MANY

Context: let ZFC* be a "rich enough finite fragment of ZFC \ Power Set Axiom. For example if, for a contradiction with Cohen we assume ZFC ⊢ $2^{\aleph_0} = \aleph_1$, this proof, since it is finite, will use only finitely many axioms of ZFC so we throw them into ZFC*. We shall a posteriori find that ZFC* will have to be chosen so that it contains more things, but then we "enrich it" at that time

like ε s in the proofs in analysis

Let, using the same logic of "a posteriori", χ be a large enough regular cardinal so that all we need in the proof happens within $(H(\chi), \in)$.
Enough of the Power Set Axiom etc.

We shall put ourselves in the following context:

M countable $\prec (H(\chi), \in)$

and M is transitive (i.e. $\in^M$ is the real $\in$)

Note ω is absolute and $M \cap \omega_1$ is an ordinal

There are plenty of such M, in fact for any $\theta \in H(\chi)$ there is such M with $\theta \in M$.

TASK AT HAND

1 Extend M to another model $M[G]$ of ZFC^*

2 $M[G]$ has the same ordinals as M

3 $M[G]$ agrees with M on the answer "is the ordinal α a cardinal?"

WE SAY $M[G]$ PRESERVES CARDINALS

4 $M[G] \models 2^{\aleph_0} \geq \aleph_2$

Then we have a contradiction with $ZFC^* \vdash 2^{\aleph_0} = \aleph_1$.

WHAT IS G?

G is a filter in a certain partial order $\mathbb{P}$.
$\mathbb{P}$ is defined in M
$\mathbb{P} = \{ p : \omega_2 \times \omega \longrightarrow 2 \mid \text{dom}(f) \text{ finite} \}$

(ω_2 ← ω_2^M)

$p \leq q$ iff $p \supseteq q$.

NOW ARGUE IN $H(\chi)$.

Lemma 5 There is a filter G on $\mathbb{P}$ such that

- for any $g \in {}^{\omega}2 \cap M = ({}^{\omega}2)^M$
  $(\forall \alpha \in \omega_2^M)(\exists p \in G)(\exists n < \omega)$
  $(\alpha, n) \in \text{dom}(p)$ & $p(\alpha, n) \neq g(n)$
- for any $n < \omega$, any $\alpha < \omega_2^M$
  $(\exists p \in G)$ $(\alpha, n) \in \text{dom}(p)$.
- if $\alpha \neq \beta$ then $(\exists p \in G)(\exists n < \omega)$
  $(\alpha, n) \in \text{dom}(p)$, $(\beta, n) \in \text{dom}(p)$
  and $p(\alpha, n) \neq p(\beta, n)$.

Proof Let $\mathcal{D}^* = \{ D \subseteq \mathbb{P} : D \text{ dense}, D \in M \}$.
In particular,

- for any $g \in {}^{\omega}2 \cap M$, for any $\alpha < \omega_2^M$
$D_{g,\alpha} = \{ p \in \mathbb{P} : \exists n \in dom(p)$
$p(\alpha, n) \neq g(n) \}$
$\in \mathcal{D}^*$

- for any $n < \omega_1$, $\alpha \in \omega_2^M$
$D_{\alpha,n} = \{ p \in \mathbb{P} : (\alpha, n) \in dom(p) \} \in \mathcal{D}^*$

- for any $\alpha \neq \beta \in \omega_2^M$
$D_{\alpha,\beta} = \{ p \in \mathbb{P} : (\exists n)\ (\alpha, n), (\beta, n) \in dom(p),$
$p(\alpha, n) \neq p(\beta, n) \} \in \mathcal{D}^*$.

By the fact that $H(\chi) \models$ "M is countable", and by a modification of the proof of Lemma 4, we know that there is a filter G such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}^*$.

Def 6 A filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for every dense set $D \subseteq \mathbb{P}$ with $D \in M$ is called $\mathbb{P}$-generic$/M$.

$G \in H(\chi)$, $G \notin M$ the proof of which will follow shortly.

Lemma 7 Suppose that G is $\mathbb{P}$-generic over M, then

- $(\forall \alpha \in \omega_2^M)$ the function $g_\alpha : \omega \to 2$ defined by $g_\alpha(n) = (\bigcup G)(\alpha, n)$ is a well defined total function $: \omega \to 2$
- if $\alpha \neq \beta \in \omega_2^M$ then $g_\alpha \neq g_\beta$
- if $g \in {}^{\omega}2 \cap M$ and $\alpha \in \omega_2^M$, then $g_\alpha \neq g$.

■

Note that to define the function g_α we only needed M and G.

Forcing Theorem 1st part | COHEN |

If $M \prec H(\chi)$ is countable, $M \models ZFC^*$ for a rich enough finite fragment of ZFC, $\mathbb{P} \in M$ is a partial order, G is $\mathbb{P}$-generic$/M$,

then there is a countable $M[G] \models ZFC^*$,

$M, G \in M[G]$,

the ordinals of M are the same as of $M[G]$. ■

SO WE HAVE ACHIEVED TASKS 1 AND 2.

however, $M[G]$ will have all the functions g_α ($\alpha \in \omega_2^M$).
If we prove that $M[G]$ preserves cardinals, TASK 3, then we have $M[G] \models 2^{\aleph_0} \geq \aleph_2$ and we are done.
(using the Completeness of First Order Logic). TASK 4

We turn a new page:

PRESERVATION OF CARDINALS

Def 8 1) A forcing notion is a partial order with the least element.

2) If $\mathbb{P}$ is a forcing notion, $p, q \in \mathbb{P}$ we say $p \perp q$ (incomparable) if $\nexists r\, (r \geq p, q)$

3) An antichain in a forcing notion is a subset A such that if $p \neq q$ are both in A, then $p \perp q$.

Def 9 A forcing notion is ccc if every antichain in it is countable.

Theorem 10 If $M, G, \mathbb{P}$ are as above, and $\mathbb{P}$ is ccc, then $M[G]$ preserves cardinals. (we say $\mathbb{P}$ preserves cardinals).

The proof depends on the second part of the Forcing Theorem which explains how $M[G]$ is obtained from M using the notion of names.

Examples of ccc forcing

1) A Cohen real

$\mathbb{P} = \{p : \omega \to 2 \mid \mathrm{dom}(p) \text{ finite}\}$

order $p \leq q$ iff $p \supseteq q$

$\mathbb{P}$ adds a "new" real, i.e. an element of ${}^{\omega}2 \setminus ({}^{\omega}2)^M$ (VG)

$\mathbb{P}$ is countable, so clearly ccc.

2) ω_2 Cohen reals

$\mathbb{P} = \{p : \omega_2 \times \omega \to 2 \mid \mathrm{dom}(p) \text{ finite}\}$,

$p \leq q$ iff $p \supseteq q$ (the order above)

$\mathbb{P}$ adds ω_2 new reals

Exercise

Prove that $\mathbb{P}$ is ccc.

Hint: use the following Δ-system lemma:

if $\{F_\alpha : \alpha < \omega_1\}$ are finite sets, then there is an uncountable $A \subseteq \omega_1$ such that there is a fixed F^* with $(\forall \alpha \neq \beta \in A)\ F_\alpha \cap F_\beta = F^*$

Similarly for other κ of the form $\kappa = \lambda^+$, $\lambda^{<\lambda} = \lambda$.

These forcings are <u>very</u> simple, yet one can prove many things just using basically these two examples, as we shall now demonstrate.