The Complexity of Recursion in Modal Logic: First Steps

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joint work with:

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12th Panhellenic Logic Symposium 26 June 2019

What this talk is about

- Two ways of Modal Logic (more, but here it's two)
- Epistemic ML: **K**, **D**, **T**, **S4**, **S5**, ...
  some restriction of Kripke structures
- Modal μ-calculus
  using fixpoints on an LTS
- Satisfiability is NP-complete, PSPACE-complete, or EXP-complete
- EML + fixpoints?

Why?

- Why not?

Why?

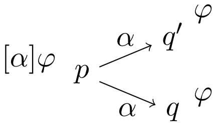
- Why not?
- EML + fixpoints may express interesting properties
- Special case: EML + versions of the μ-calculus, which we hope can express *monitorability* requirements

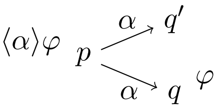
EML with fixpoints

$$\varphi, \psi \in L ::= \texttt{tt} \mid \texttt{ff} \mid p \mid \neg p \mid X \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \langle \alpha \rangle \varphi \mid [\alpha]\varphi \mid \mu X.\varphi \mid \nu X.\varphi$$

We interpret formulae on states from an LTS:

$$\langle \textsc{Proc}, \textsc{Act}, \rightarrow, V \rangle$$





$\nu X.\varphi(X)$ is a maximum fixed point:
$\nu X.\varphi \equiv \varphi[\nu X.\varphi/X]$ and is true if there is no reason to be false

$\mu X.\varphi(X)$ is a minimum fixed point:
$\mu X.\varphi \equiv \varphi[\mu X.\varphi/X]$ and is true if there is a reason to be true

LTS restrictions

- *D*: $\xrightarrow{\alpha}$ is serial;
- *T*: $\xrightarrow{\alpha}$ is reflexive — $\forall s.\ s \xrightarrow{\alpha} s$;
- 4: $\xrightarrow{\alpha}$ is transitive — $\forall s, t, r.\ (s \xrightarrow{\alpha} t \xrightarrow{\alpha} r \;\Rightarrow\; s \xrightarrow{\alpha} r)$;
- 5: $\xrightarrow{\alpha}$ is Euclidean — $\forall s, t, r.$ if $s \xrightarrow{\alpha} t$ and $s \xrightarrow{\alpha} r$, then $t \xrightarrow{\alpha} r$.

$\mathbf{D}_k = \mathbf{K}_k + D$, $\mathbf{T}_k = \mathbf{K}_k + T$, $\mathbf{K4}_k = \mathbf{K}_k + 4$, $\mathbf{D4}_k = \mathbf{K}_k + D + 4$, $\mathbf{S4}_k = \mathbf{K}_k + T + 4$, $\mathbf{KD45}_k = \mathbf{D4}_k + 5$, $\mathbf{S5}_k = \mathbf{S4}_k + 5$.

$\mathbf{L}^\mu = \mathbf{L}$ + fixpoints

$\mathbf{K}^\mu$ is the μ-calculus

Common Knowledge

everyone knows that everyone knows that…

it's a fixpoint for [everyone][everyone][everyone] $\cdots \varphi$

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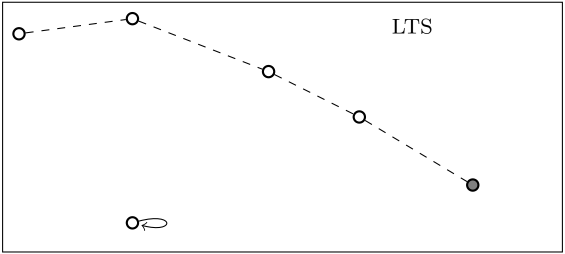

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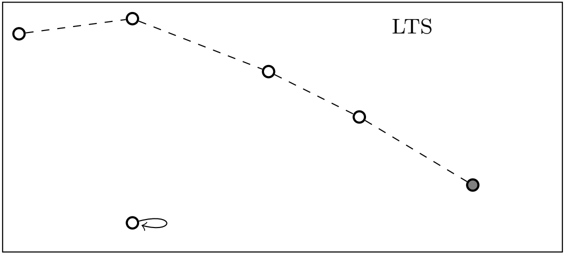

$$C\varphi := \nu X.[\text{everyone}](\varphi \wedge X)$$

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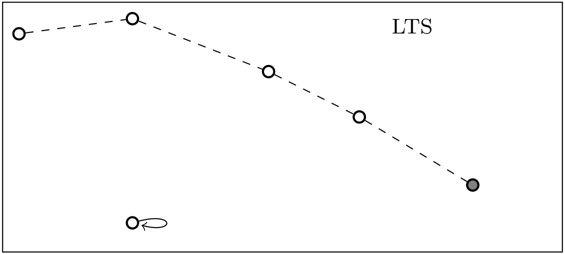

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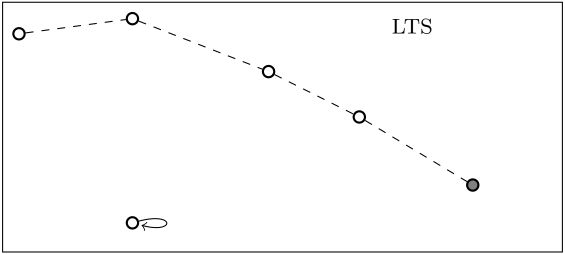

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There is rumor that φ $\mu X.(\bigvee_\alpha [\alpha]\varphi \vee \bigvee_\alpha [\alpha]X)$

EXP

Theorem (Kozen, 1983)

Satisfiability for $\mathbf{K}_k^\mu$ *is* EXP*-complete.*

Theorem (Halpern & Moses, 1992)

Satisfiability for $\mathbf{L}_k + C$, $k > 1$ *is* EXP*-complete.*

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Satisfiability for $\mathbf{D}_k^\mu$, $\mathbf{K4}_k^\mu$, *and* $\mathbf{D4}_k^\mu$ *is in* EXP.

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Proof.

$C(\bigvee_{\alpha \in \textsc{Act}} \langle \alpha \rangle \texttt{tt})$ imposes seriality; $\nu X.[i](\varphi \wedge X)$ simulates transitivity. □

Negative Introspection, One Action

Theorem (Ladner, 1977; Halpern & Rêgo, 2007)

If $\mathbf{L} \in \{\mathbf{K}, \mathbf{T}, \mathbf{D}, \mathbf{K4}, \mathbf{D4}, \mathbf{S4}\}$, *then* $\mathbf{L}$*-satisfiability is* PSPACE*-complete and* $\mathbf{L} + 5$*-satisfiability is* NP*-complete.*

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If $\mathbf{L} \in \{\mathbf{K}, \mathbf{T}, \mathbf{D}, \mathbf{K4}, \mathbf{D4}, \mathbf{S4}\}$, *then and* $\mathbf{L}^{\mu} + 5$*-satisfiability is* NP*-complete.*

With similar techniques.

Open

- One action, no negative introspection
- Constraints T and 5 on many actions
- Axiomatizations, completeness
- What other interesting things can we say with belief and fixpoints?

	$k = 1$	$k > 1$
K	EXP-complete	EXP-complete
at most 4, D	in EXP	EXP-complete
also T	PSPACE-hard	EXP-hard
with 5	NP-complete	EXP-hard

The End

Time for questions

Thank you for your attention

This research was supported by the projects "TheoFoMon: Theoretical Foundations for Monitorability" (grant number: 163406-051) and "Epistemic Logic for Distributed Runtime Monitoring" (grant number: 184940-051) of the Icelandic Research Fund.